Self-adjoint codimension 2 boundary conditions for Dirac operators

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Dirac operators in topology, geometry
and representation theory
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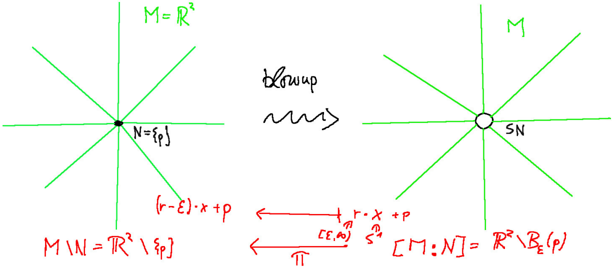
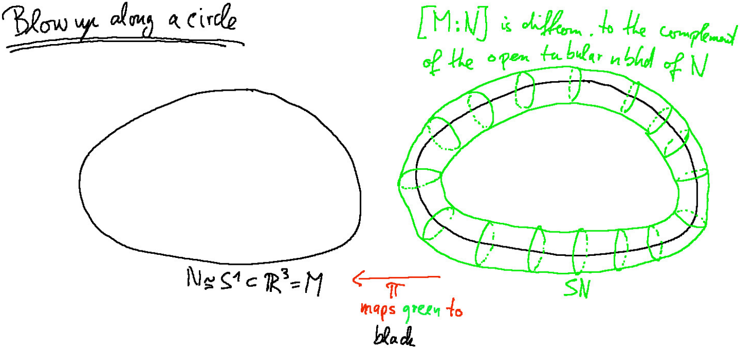
The setting of this talk

- Let (M, g) be a complete oriented Riemannian manifold, N a compact oriented submanifold of codimension k.
- $[M : N] = (M \setminus N) \cup S_M N$ the blowup of M along N. Here $S_M N$ is the normal sphere bundle of N in M, $S_M N = \partial[M : N]$.



The setting of this talk. Page 2



The pull-back $\hat{g}|_p = (\pi^* g)|_p : T_p[M:N] \otimes T_p[M:N] \to \mathbb{R}$ is degenerate along the fibers of $S_M N \to N$.

The setting of this talk. Page 3

- We assume that $M \setminus N$ is spin. Thus there is a complex spinor bundle $\Sigma \to [M : N]$.
  Clifford multiplication

$$T_p[M : N] \otimes \Sigma_p \to \Sigma_p, \quad X \otimes \varphi \mapsto X \cdot \varphi$$

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi + 2\hat{g}(X, Y)\varphi = 0.$$

  If the spin structure extends to M, then $\Sigma = \pi^*(\Sigma M)$.
- Let $L \to [M : N]$ be a hermitian line bundle with ∇, whose curvature is a pull-back from M.
  $R^\nabla = i\pi^*\alpha$, $\alpha \in \Gamma(\bigwedge^2 T^*M)$.
- $W := \Sigma \otimes L$ generalized spinor bundle on $[M : N]$

More general frameworks are possible which will not be discussed in this talk.

Examples with different codimensions

- $\dim N = \dim M - 1$: Classical boundary problem.
  If N separates M in M_1 and M_2, then

$$[M : N] = (M_1 \cup N) \amalg (M_2 \cup N).$$

  No degeneracy!

- $\dim N = \dim M - 2$. Monodromy $\alpha = (\alpha_1, \dots, \alpha_j)$.

$$N = \coprod_{j=1}^{\ell} N_j$$

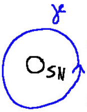
  Parallel transport in W around N_j is $e^{2\pi i \alpha_j}$.
  $[\alpha_j] \in \mathbb{R}/\mathbb{Z}$ only depends on j.
  Main objective of the talk.

- $\dim N \leq \dim M - 3$.
  Then $L = \pi^*(\mathcal{L})$. No monodromy effects.
  Furthermore N is "invisible".

Monodromy

Monodromie around N

Parallel transport in Σ close to SN is $\pm$ id



Parallel transport along γ

Parallel transport in L is $e^{2\pi i \hat{\alpha}_j}$, $\hat{\alpha}_j \in \mathbb{R}$ konstant along connected components of N

$N = \{p\} \subset \mathbb{R}^2 = M$

$\Rightarrow$ Parallel transport in $W = \Sigma \otimes L$ is $e^{2\pi i \alpha_j}$, $\alpha_j = \begin{cases} \hat{\alpha}_j \\ \hat{\alpha}_j + \frac{1}{2} \end{cases}$

Main examples

- M spin. Monodromy comes from L.
  Main subcase: L flat. Monodromy $\pi_1(M \setminus N) \to S^1$.
  Main subsubcase: N is a link in $\mathbb{S}^3$.

$$(S^1)^\ell \ni \exp 2\pi i\alpha \mapsto L_\alpha$$

- $(M \setminus N) \cup N_j$ is spin. Similar discussion close to N_j

Main examples, ct'd

- $(M \setminus N) \cup N_j$ is not spin, (more precisely: spin structure does not extend).
  Then monodromy only comes from Σ, $\alpha_j = 1/2 \mod \mathbb{Z}$.
  This case was treated by Degeratu & Stern where they showed the importance of the minimal and maximal extension, the importance of self-adjoint extensions, and relations to PMT, and several of the effects described below in this special case.
  Main subcase: $L = \underline{\mathbb{C}}$
  Example: $M = \mathbb{C}P^{2r}$, $N = \mathbb{C}P^{2r-1}$.
  Fix $p \in M \setminus N$, solve $\not{D}\Psi = \psi_0 \delta_p$ on $M \setminus N$ with bdy cond.
  Expectation: If PMT would fail, we would get a map

  $$S(\Sigma_p) \times \{\text{bdy cond}\} \to \{\text{non-zero spinors on } N\}.$$

  Interesting applications?

Genesis of the project

Work by mathematical physicists for $M = \mathbb{S}^3$ or $M = \mathbb{R}^3$.
Electrons coupled magnetic fields.
Existence of harmonic spin^c-spinors yield statements of the type

> If our world is stable, then the fine structure constant $\hbar c/e^2$ has to satisfy some bounds.

Measurements: $\hbar c/e^2 = 137.03599968\ldots$. Why this?

Examples of harmonic spin^c-spinors on $M = \mathbb{S}^3$ with distributional magnetic field α along N yield smooth solutions on $\mathbb{R}^3$: smoothing of magnetic field, conformal change.

Leads to link invariants, Hopf insulators (3-d topological insulators)

Some literature (incomplete!)

- Aharonov & Casher 1978: general description
- Loss & Yau (& Fröhlich) 1986: first examples of harmonic spinors, relation to "stablity of matter" and "estimates of the fine structure constant"
- László Erdös & Solovej 2001: good progress, examples with many harmonic spinors on $\mathbb{S}^3$, sketchy
- Portmann & Sok & Solovej 2015–2018: mathematically profound, but e.g. no flat complex line bundles are used. Spinors $S^3 \to \mathbb{C}^2$ are glued along Seifert surfaces
- Lieb & Seiringer 2010. Book "Stability of matter". Much broader, mathematically rigorous, interesting to read

Questions

Is this mathematically rigorous?

Interesting consequences for knot theory?

Interesting new boundary conditions for new applications?

My perspective

- Boris Botvinnik and Nikolai Saveliev asked me: can we rigorously follow the calculation of these knot invariants? More information about them? Interesting discussions(delayed by Corona ... continued in Cortona)
- joint project with Nadine Große: classification of the self-adjoint extensions in the general setting
- next steps: boundary regularity, compact resolvents, Fredholmness, index theory, KO-theoretical framework

Disclaimer: Work **in progress.** Still sign mistakes, $\mathrm{l.o.t.}$-terms neglected etc.. Some parts will be sketchy.

Other mathematical literature?

The problem can be interpreted as a stratified space with strata of dimensions *m* and $m - 2$. Much literature, but our case does not seem to be covered (at least in 2020)

- Albin & Gell-Redman 2016: incomplete edge space. Self-adjoint extensions, Fredholmness, index theory. This seems to fit. However, A&G-R require a spectral condition, called “Witt condition” which is in our case only satisfied for $\alpha \in \mathbb{Z}^\ell$.
- Mazzeo: has work prior to A&G-R on a blown-up version, seems to have gone into A&G-R
- Krainer & Mendoza
- Leichtnam & Mazzeo & Piazza
- Newer work by Albin & Piazza and coauthors including spaces without Witt condition, probably see talk by Pierre Albin today.
- Brüning, Sergiu Moroianu, Atiyah & LeBrun

Our approach is to provide a reference in the style of the Bär-Ballmann approach to codimension 1-boundary conditions.

Self-adjoint extensions. Again: the setting

(joint work in progress with Nadine Große, Freiburg)

- N a compact oriented submanifold of codimension 2 of M.
- $\pi : [M : N] \to M$ the blowup of M along N.
  $S_M N = \partial[M : N] = \pi^{-1} N$.
  $\hat{g} = \pi^* g : T_p[M : N] \otimes T_p[M : N] \to \mathbb{R}$ is degenerate along circle fibers of $S_M N \to N$
- $W \to [M : N]$ a suitable generalized $\hat{g}$ spinor bundle

$$N = \coprod_{j=1}^{\ell} N_j$$

Monodromy $\alpha = (\alpha_1, \ldots, \alpha_\ell)$.
Parallel transport in W around N_j is $e^{2\pi i \alpha_j}$.

The associated Dirac operator $\not{D}$ is a formally self-adjoint 1st order differential operator.

Minimal and maximal closed extensions

$C_c^\infty([M:N],W) :=$
$\{$sections of W with compact support in $[M:N]\}$

$C_{cc}^\infty([M:N],W) :=$
$\{$sections of W with compact support in $M \setminus N\}$

The minimal Dirac operator $\not{D}_{\min}$ is the Dirac operator whose domain is the closure of $C_{cc}^\infty(W)$ with respect to the graph norm

$$\|\varphi\|_D^2 := \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2.$$

$\not{D}_{\min}$ is symmetric.

$\not{D}_{\max} := \not{D}_{\min}^*$, symmetry implies $\mathrm{dom}(\not{D}_{\min}) \subset \mathrm{dom}(\not{D}_{\max})$.

Our Goal: Find all domains $\mathcal{D}$ with
$\mathrm{dom}(\not{D}_{\min}) \subset \mathcal{D} \subset \mathrm{dom}(\not{D}_{\max})$ such that

$$\not{D}_{\max}|_{\mathcal{D}}$$

is self-adjoint.

The role of $C_c^\infty([M:N], W)$

For codimension 1 boundaries: $C_c^\infty([M:N], W)$ is dense in $\mathrm{dom}(\not{D}_{\max})$.
Is this true for codimension 2 as well?

No! Then $\mathrm{dom}(\not{D}_{\max})$ is **not** the closure of $C_c^\infty([M:N], W)$.

Problem: $\not{D} : C_c^\infty([M:N], W) \to C_c^\infty([M:N], W)$ not defined.

Even worse: $\not{D}(\varphi|_{M\setminus N}) \notin L^2$, unless if φ is parallel along the circles of $S_M N \to N$.

Other approaches to describe $\mathrm{dom}(\not{D}_{\max})$ in terms of $C_c^\infty([M:N], W)$ failed.

The case $\alpha \in \mathbb{Z}^\ell$

In this case $W = \pi^*(\mathcal{W})$.

Lemma 1.

Let M be a complete manifold with generalized spinor bundle $\mathcal{W}$. Let $\mathrm{H}^1_{\not{D}}(M, \mathcal{W})$ be the completion of $C^\infty_c(M, \mathcal{W})$ w.r.t. the graph norm of $\not{D}$. If $N \subset M$ is (a compact submanifold) of codimension ≥ 2, then $C^\infty_c(M \setminus N, \mathcal{W}) = C^\infty_{cc}([M : N], \mathcal{W})$ is dense in $\mathrm{H}^1_{\not{D}}(M, \mathcal{W})$.

Thus: "*N* is invisible."

Lemma 1.

Let M be a complete manifold with generalized spinor bundle $\mathcal{W}$. Let $\mathrm{H}^1_{\not D}(M, \mathcal{W})$ be the completion of $C^\infty_c(M, \mathcal{W})$ w.r.t. the graph norm of $\not D$. If $N \subset M$ is (a compact submanifold) of codimension ≥ 2, then $C^\infty_c(M \setminus N, \mathcal{W}) = C^\infty_{cc}([M : N], \mathcal{W})$ is dense in $\mathrm{H}^1_{\not D}(M, \mathcal{W})$.

Proof.

Wlog codimension 2.
Let $\varphi \in C^\infty_c(M, \mathcal{W})$.
Take a logarithmic cut-off

$$\chi_{k,\epsilon}(x) := \begin{cases} 0 & \text{for } r(x) \leq e^{-k}\epsilon\,, \\ \frac{1}{k}\log\frac{r(x)\,e^k}{\epsilon} & \text{for } e^{-k}\epsilon \leq r(x) \leq \epsilon\,, \\ 1 & \text{for } r(x) \geq \epsilon\,. \end{cases} \tag{1}$$

Then

$$\left\|\nabla(\chi_{k,\epsilon}\varphi) - \nabla\varphi\right\|_{L^2} \leq C(\epsilon + \sqrt{k}). \tag{2}$$

For $\epsilon = k^{-1/2} \to 0$ we have $\chi_{k,\epsilon}\varphi \to \varphi$. □

Some positive results (without proofs)

Lemma.
Suppose that $\varphi \in \mathrm{dom}(\not{D}_{\max})$ *is bounded on a neighbourhood of* N*. Then* $\varphi \in \mathrm{dom}(\not{D}_{\min})$*.*

Lemma.
Assume that the geometry of g *and* W *is bounded,* $\not{D}$ *coercive at infinity. Then on dom(*$\not{D}_{\min}$*) the graph-norm for* $\not{D}$ *is equivalent to the classical* H^1*-norm, i.e. the graph norm for* ∇*.*

Lemma.
For an L^1_{loc}*-section* φ *of* W *we define* $D\varphi$ *in the distributional sense where as test functions we use the compactly supported smooth sections of* $W^* \otimes \bigwedge^n T^*M$*. Then* $\mathrm{dom}(\not{D}_{\max})$ *is the vector space of all* L^1_{loc}*-section of* W *for which* φ *and* $D\varphi$ *are in* L^2*.*

Abstract extension space

$$\check{Q} := \frac{\text{dom } \not{D}_{\max}}{\text{dom } \not{D}_{\min}}$$

abstract extension space with graph norm.
For $\varphi, \psi \in \text{dom}(\not{D}_{\max})$ we define

$$\check{b}([\varphi],[\psi]) := \int_{M \setminus N} \Big(\langle \not{D}\varphi, \psi \rangle - \langle \varphi, \not{D}\psi \rangle \Big) \, dv^g .$$

It is a well-defined, non-degenerate skew-hermitian form on $\check{Q}$.
Goals:
Identify this as $\check{\mathrm{H}}$-sections of a bundle over N.
Show that the pairing is perfect.
$\{\text{self-adj. bdy cond.}\} \stackrel{1:1}{\longleftrightarrow} \{\text{Lagrangian subspaces of } (\check{Q}, \check{b})\}$

The normal volume element

Let (e_1, e_2) be a positively oriented orthornormal frame of the normal bundle $\nu_M N$ at p.
We define $\omega_{\mathrm{nor}} := e_1 \cdot e_2 \in \mathrm{End}(W_p)$.
Extend smoothly for p in neigborhood of N. Decompose into ω_{nor}-eigenspace bundles for eigenvalues $\pm i$.

$$W = W_+ \oplus W_-$$

$$D\!\!\!/ = \underbrace{D\!\!\!/^{\mathrm{nor}}}_{\text{odd}} + \underbrace{\partial_r \cdot D\!\!\!/^{N}}_{\text{even}} + \mathrm{l.o.t.}$$

Portman-Sok-Solovej boundary conditions

Choose a sign $\epsilon_j \in \{\pm 1\}$ for each $j = 1, \ldots, \ell$.
Close to N_j the boundary condition is

$$\mathcal{B} = \{\varphi \in \mathrm{dom}(\not{D}_{\max}) \mid (\omega_{\mathrm{nor}} + i\epsilon_j) \cdot \varphi \in \mathrm{dom}(\not{D}_{\min})\}.$$

Theorem 2 (PSS ≈ 2017).
This is a self-adjoint boundary condition in the case $M = \mathbb{S}^3$, N a link, L flat.

We extend this the whole setting, but there are many more self-adjoint extensions.

Continuity in α

Is the PSS boundary condition continuous in α?

The PSS boundary condition is

- continuous for $\epsilon_j \alpha_j \nearrow 0 \mod \mathbb{Z}$,
- but non-continuous for $\epsilon_j \alpha_j \searrow 0 \mod \mathbb{Z}$.

General boundary conditions

$\{$self-adj. bdy cond.$\} \overset{1:1}{\longleftrightarrow} \{$Lagrangian subspaces of $(\check{Q}, \check{b})\}$
$\check{Q} = \check{\mathrm{H}}^{\beta,\gamma}(V_\alpha)$ for some bundle $V_\alpha \to N$.
The $\check{\mathrm{H}}$-spaces have a both-sided regularity incontinuity at
$\alpha_j \equiv 1/2 \mod \mathbb{Z}$

Importance of continuity

Spectral flow arguments
Fredholm index is not constant at $\alpha_j \equiv 0 \mod \mathbb{Z}$.

2-dimensional model space

Assume $M = \mathbb{C} \ni z$, $N = \{0\}$, $\Sigma = \underline{\mathbb{C}^2} = \Sigma_+ \oplus \Sigma_-$
L flat bundle over $[\mathbb{C} : \{0\}]$, monodromy α
Then ω_{nor} is the standard volume element.

$$\not{D} = \not{D}^{\mathrm{nor}} = \sqrt{2} \begin{pmatrix} 0 & \overline{\partial} \\ -\partial & 0 \end{pmatrix}$$

$\frac{z^{-\alpha}}{|z|^{-\alpha}}$ represents a nowhere vanishing smooth section of L.
Ansatz:

$$\Phi^+_{\beta,\gamma} := \begin{pmatrix} z^\beta \overline{z}^\gamma \\ 0 \end{pmatrix}, \qquad \Phi^-_{\beta,\gamma} := \begin{pmatrix} 0 \\ z^\beta \overline{z}^\gamma \end{pmatrix}.$$

where β and γ over real numbers with $\beta - \gamma + \alpha \in \mathbb{Z}$.
$\Phi^\pm_{\beta,\gamma} \in L^2_{\mathrm{loc}}$ iff $\beta + \gamma > -1$

$$\not{D}\Phi^+_{\beta,\gamma} = -\sqrt{2}\beta\Phi^-_{\beta-1,\gamma}, \qquad \not{D}\Phi^-_{\beta,\gamma} = \sqrt{2}\gamma\Phi^+_{\beta,\gamma-1},$$

Lemma.

The condition that $\Phi^{\pm}_{\beta,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *is characterized as follows ("locally around* 0*").*

(1) *Suppose* $\beta \neq 0$ *and* $\gamma \neq 0$*. Then* $\Phi^{\pm}_{\beta,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only* $\beta + \gamma > 0$*.*

(2) *Suppose* $\beta = 0$ *and* $\gamma \neq 0$*. Then* $\Phi^{+}_{0,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\gamma > -1$*, and* $\Phi^{-}_{0,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\gamma > 0$*.*

(3) *Suppose* $\beta \neq 0$ *and* $\gamma = 0$*. Then* $\Phi^{+}_{\beta,0} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\beta > 0$*, and* $\Phi^{-}_{\beta,0} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\beta > -1$*.*

(4) *Suppose* $\beta = \gamma = 0$*.* $\Phi^{\pm}_{0,0} \in \mathrm{dom}(\not{D}_{\max}) = \mathrm{dom}(\not{D}_{\min})$*.*

$\alpha \in (0,1)$: Then elements in $\mathrm{dom}(\not{D}_{\max})$ are of the form

$$\begin{pmatrix} \bar{z}^{\alpha-1}\varphi_+ \\ z^{-\alpha}\varphi_- \end{pmatrix} + \mathrm{dom}(\not{D}_{\min}).$$

Higher dimensions: Extension map and Trace map

Again codimension 1.
The restriction map $\mathcal{R} : C_c^\infty(M; W) \to C_c^\infty(\partial M; W)$ extends to a continuous map, called "trace map",

$$\mathcal{R} : \mathrm{dom}(\not{D}_{\max}) \to H^{-1/2}(\partial M; W)$$

However this is **not surjective.** $\check{H}(\partial M; W) := \mathcal{R}\big(\mathrm{dom}(\not{D}_{\max})\big)$.
Decompose

$$C_c^\infty(\partial M; W) = \mathcal{S}_+ \oplus \mathcal{S}_- .$$

Obtain $\check{H}(\partial M; W)$ by completing $\mathcal{S}_+$ with respect to the $H^{1/2}$-norm and $\mathcal{S}_-$ with respect to the $H^{-1/2}$-norm.
There is a continuous extension map

$$\mathcal{E} : \check{H}(\partial M; W) \to \mathrm{dom}(\not{D}_{\max}), \qquad \mathcal{R} \circ \mathcal{E} = \mathrm{id}\,.$$

Idea: Similar approach in codimension 2?

Higher dimensions: Extension map and Trace map

Back to codimension 2.

Now: For simplicity of presentation let N be connected.

Idea: The trace map is given by

$$\begin{aligned} \mathcal{R} : \mathrm{dom}(\not{D}_{\max}) &\to \Gamma(W|_{S_M N}) \\ \varphi &\mapsto \lim_{r \searrow 0} \begin{pmatrix} r^{1-\alpha} & 0 \\ 0 & r^{\alpha} \end{pmatrix} \varphi|_{\partial U_r(N)} \end{aligned}$$

$$\begin{aligned} \check{b}([\varphi],[\psi]) &\overset{\text{def}}{=} \int_{M \setminus N} \Big(\langle \not{D}\varphi, \psi \rangle - \langle \varphi, \not{D}\psi \rangle \Big)\, dv^g \\ &= B\big(\mathcal{R}(\varphi), \mathcal{R}(\psi)\big) \end{aligned}$$

where $B(\Phi, \Psi) = \int_{S_M N} \langle \Phi, \partial_r \cdot \Psi \rangle \, d\mu$ and where μ is the S^1-equivariant measure on $S_M N$ with $\pi_* \mu = 2\pi \, \mathrm{dvol}^N$.

We obtain:

$\check{\mathrm{H}}^{\beta,\gamma}(V_\alpha) := \mathrm{Image}(\mathcal{R}) \subset \check{\mathrm{H}}(W|_{S_M N})$
Extension operator

$$\mathcal{E} : \check{\mathrm{H}}^{\beta,\gamma}(V_\alpha) = \mathrm{Image}(\mathcal{R}) \to \mathrm{dom}(\not{D}_{\max})$$

Properties:

$$\begin{aligned} \mathcal{R} \circ \mathcal{E} &= \mathrm{Id} \\ \check{b}(\varphi, \mathcal{E}(\Psi)) &= B(\mathcal{R}(\varphi), \Psi) \end{aligned}$$

B is a perfect pairing on $\check{\mathrm{H}}^{\beta,\gamma}(V_\alpha)$.

To determine $\check{\mathrm{H}}^{\beta,\gamma}(V_\alpha)$ we have to consider

- S^1-equivariance
- regularity along N

Equivariance

Let $\alpha \in (0,1)$
$S^1 \subset \mathbb{C}$ acts on the S^1-principle bundle $S_M N \to N$:
$\rho : S^1 \to \mathrm{Diff}(S_M N)$.
Then $K := d\rho(i)$ a vector field on $S_M N$.
We define

$$\begin{aligned}
\Gamma_\alpha(W^+|_{S_M N}) &:= \left\{\Phi \in C^\infty(W^+|_{S_M N}) \text{ with } \nabla_K \Phi = i(1-\alpha)\Phi\right\} \\
\Gamma_\alpha(W^-|_{S_M N}) &:= \left\{\Phi \in C^\infty(W^-|_{S_M N}) \text{ with } \nabla_K \Phi = -i\alpha\Phi\right\} \\
\Gamma_\alpha(W|_{S_M N}) &:= \Gamma_\alpha(W^+|_{S_M N}) \oplus \Gamma_\alpha(W^-|_{S_M N})
\end{aligned}$$

$\Gamma_\alpha(W|_{S_M N})$ is the space of sections of a vector bundle $V_\alpha \to N$.

$$\Gamma_\alpha(W|_{S_M N}) = \Gamma(V_\alpha)$$

Density and regularity

Let $\Phi_\pm \in \Gamma_\alpha(W^\pm|_{S_M N})$.
Then

$$\chi(r)\left(r^{\alpha-1}\Phi_+ + r^{-\alpha}\Phi_-\right) \in \mathrm{dom}(\not{D}_{\max}).$$

Up to l.o.t. and $\nabla\chi$-terms it is in the kernel of the normal Dirac operator $\not{D}^{\mathrm{nor}}$.

$\Gamma_\alpha(W|_{S_M N}) = \Gamma(V_\alpha)$ is dense in the Hilbert space $\check{\mathrm{H}}^{\beta,\gamma}(V_\alpha)$.

To explain the norm on the space we will discuss

- The canonical metric on the normal bundle
- The N-Dirac operator
- The corresponding $\check{\mathrm{H}}$-spaces

Canonical metric on the normal bundle

To understand codimension 1 boundary conditions, one has to understand half-cylinders $N \times [0, \infty)$ first.
In fact, half cylinders are a special case of the (blown-up) canonical metric on the normal bundle.

Let $N \subset M$ be of codimension k. The canonical metric is a Riemannian metric on the total space of $\pi : \nu_M N \to N$ such that

- π is a Riemannian submersion,
- the horizontal spaces $\mathcal{H}_p$ are given by the connection on $\nu_M N \to N$,
- for $V \in \nu_M$ the vertical space in V is naturally isometric to $\nu_M N|_{\pi(V)}$.

The Dirac operator $\not{D}_0$ on $(\nu_M N, g_{\mathrm{can}})$ is our **model operator.**

The N-Dirac operator

The horizontal spaces also define a distribution $\mathcal{H}$ of codimension $k-1$ in $S_M N$.
For an onb $e_1, \dots, e_{m-k}$ of $\mathcal{H}_p$ and $\varphi \in \Gamma(W|_{S_M N})$ we define the N-Dirac operator as

$$\left(\not{D}^N \varphi\right)|_p := -\sum_{j=1}^{m-k} \partial_r \cdot e_j \cdot \nabla_{e_j} \varphi.$$

Recall $\not{D} = \not{D}^{\mathrm{nor}} + \partial_r \cdot \not{D}^N + \mathrm{l.o.t.}$

Lemma.
The operator $\not{D}^N$ is an odd, formally self-adjoint, elliptic operator of Dirac type on N.

Back to our codimension 2 setting

On the model space we have

$$\not{D}_0 = \underbrace{\partial_r \cdot \nabla_r + \frac{K}{r} \cdot \nabla_{K/r}}_{\not{D}^{\mathrm{nor}}} + \partial_r \cdot \not{D}^N$$

$$= \partial_r \cdot \left(\nabla_r - \omega_{\mathrm{nor}} \cdot \nabla_{K/r} + \not{D}^N \right)$$

Note that

$$\left(\nabla_r - \omega_{\mathrm{nor}} \cdot \nabla_{K/r} \right) \left(r^{\alpha - 1} \varphi_+ \right) = 0$$

$$\left(\nabla_r - \omega_{\mathrm{nor}} \cdot \nabla_{K/r} \right) \left(r^{-\alpha} \varphi_- \right) = 0$$

Idea: Analyse this in a spectral decomposition for $\not{D}^N$.
This will give us the Ȟ-space.

The $\check{\mathrm{H}}_\alpha$ spaces

Theorem 3.

Let $\alpha \in (0,1)$. *We obtain a splitting*

$$\begin{aligned} \Gamma_\alpha(W|_{S_M N}) &= V_+ \oplus V_- \\ \check{\mathrm{H}}_\alpha(W|_{S_M N}) &= \overline{V_+}^{H^\beta} \oplus \overline{V_-}^{H^{-\beta}} \end{aligned}$$

where $\beta := \min\{\alpha, 1-\alpha\}$.
There is a surjective trace map $\mathcal{R} : \mathrm{dom}(\not{D}_{\max}) \to \check{\mathrm{H}}_\alpha(W|_{S_M N})$ *with kernel* $\mathrm{dom}(\not{D}_{\min})$ *and an injective extension map* $\mathcal{E} : \check{\mathrm{H}}_\alpha(W|_{S_M N}) \to \mathrm{dom}(\not{D}_{\max})$ *with*

$$\begin{aligned} \mathcal{R} \circ \mathcal{E} &= \mathrm{Id} \\ \check{b}(\varphi, \mathcal{E}(\Psi)) &= B(\mathcal{R}(\varphi), \Psi) \end{aligned}$$

B is a perfect pairing on $\check{\mathrm{H}}_\alpha(W|_{S_M N})$.

$V_- := \{\Phi \in \Gamma_\alpha(W|_{S_M N}) \mid \Phi$ "extends" to a $\not{D}_0$-harmonic L^2-spinor$\}$

The Ansatz

Attention: $\not{D}^N$ anticommutes with ω_{nor}.
We assume $\not{D}^N \Phi = \lambda \Phi$, $\Phi = (\Phi_+, \Phi_-)$.
For $r \to \infty$: $\not{D}^N$ dominates, thus $L^2 \Leftrightarrow \lambda > 0$
For $r \to 0$: $\nabla_{K/r}$ dominates

Ansatz

We search for a solution asymptotic to $\exp(-\lambda r)\Phi$

$$\varphi = f_+(r)\Phi_+ + f_-(r)\Phi_-, \qquad f = (f_+, f_-)$$

$\not{D}_0 \varphi = 0$ then translates into

$$0 = f'(r) + \frac{1}{r} \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix} f(r) + \lambda f(r)$$

The asymptotics for $r \to 0$ of solutions of this ODE depend strongly on the sign of $\alpha - \frac{1}{2}$.

The concrete extension spaces $\check{\mathsf{H}}_\alpha$

For $\alpha \in (0, 1/2)$: for a smooth section $\Phi = (\Phi_+, \Phi_-)$ of $W|_{\mathbb{S}_M N}$

$$\|\Phi\|^2_{\check{\mathsf{H}}} := \|\Phi_+\|^2_{\mathsf{H}^{-\alpha}} + \|\Phi_-\|^2_{\mathsf{H}^{\alpha}}$$

For $\alpha \in (1/2, 1)$: for a smooth section $\Phi = (\Phi_+, \Phi_-)$ of $W|_{\mathbb{S}_M N}$

$$\|\Phi\|^2_{\check{\mathsf{H}}} := \|\Phi_+\|^2_{\mathsf{H}^{1-\alpha}} + \|\Phi_-\|^2_{\mathsf{H}^{\alpha-1}}$$

For $\alpha = 1/2$: the space V_- is spanned by the eigenspinors of $\not{D}^N$ to the positive eigenvalues.

The extension map

On V_- it is obtained by solving the ODE backwards: from $r \to 0$ to $r \to \infty$.

$$V_- \to \mathrm{dom}(\not{D}_{\max})$$

What do we do with V_+? (for simplicity $\alpha \neq 1/2$)
Extend $\Phi \in V_+$ by

$$\mathcal{E}(\Phi) := r^{\beta-1} \exp\left(-|\not{D}^N| r\right) \Phi.$$

Then $\not{D}_0 \varphi \neq 0$, but the L^2-norm of $\not{D}_0 \varphi$ remains sufficiently well-controlled.

Why is it impossible to find an extension on a larger space $\widetilde{\mathsf{H}}$? Why is it impossible that $\operatorname{Image}\mathcal{R}$ is larger?

(Until now we only have seen arguments for $\check{\mathsf{H}}_\alpha \subseteq \operatorname{Image}\mathcal{R}$!)

Answer: As we have found a space, on which B is a perfect pairing!
Consider the continuous map

$$\Psi \mapsto b(\varphi, \mathcal{E}(\Psi)) = B(\mathcal{R}(\varphi), \Psi)$$

Thus $B(\mathcal{R}(\varphi), \bullet) \in \widetilde{\mathsf{H}}^*$

$$\Longrightarrow \quad \mathcal{R}(\varphi) \subseteq \widetilde{\mathsf{H}}^{*B} \subseteq \check{\mathsf{H}}^{*B} = \check{\mathsf{H}}.$$

So, if $\check{\mathsf{H}} \subsetneq \widetilde{\mathsf{H}}$ is a strict inclusion, then $\widetilde{\mathsf{H}}^{*B} \subsetneq \check{\mathsf{H}}$, thus we get a contradiction to $\mathcal{R} \circ \mathcal{E} = \mathrm{Id}$.

Summary

For *N* connected, codimension 2

- $\alpha \in \mathbb{Z}$: $\mathrm{dom}(\not{D}_{\max}) = \mathrm{dom}(\not{D}_{\min}) = \mathrm{dom}(D^M)$
- For each $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ selfadjoint extensions are in bijection to Lagrangian closed subspaces of $\check{H}_\alpha$
  $\rightsquigarrow$ Description of all self-adjoint extensions.
- Positive PSS boundary conditions continuous for $\alpha \searrow 0$.
  Negative PSS boundary conditions continuous for $\alpha \nearrow 0$.