Spin^c-manifolds, positive scalar curvature and manifolds with singularities

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Dirac Operators in Topology, Geometry and Representation Theory

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This work is a part of the joint projects with P. Piazza and J. Rosenberg.

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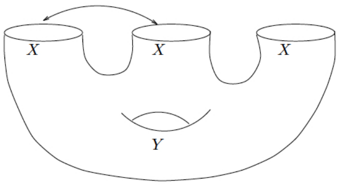
1. Some Old Results. Let (L, g_L) be a closed Riemannian manifold, where g_L zero scalar curvature. Then let Y be a closed smooth manifold, such that the product $Y \times L$ is a boundary of a smooth manifold X, i.e. $\partial X = Y \times L$.



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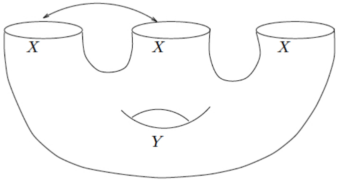

Naive Question: *Does there exist a psc-metric g_Y on Y, such that the metric $g_Y + g_L$ on $\partial X = Y \times L$ can be extended (being a product near ∂X) to a psc-metric g_X on X?*

Denote $\beta X := Y$. Let $C(L)$ be a cone over L. Then

$$X_\Sigma := X \cup_{\partial X} -(\beta X \times C(L)).$$

is a *manifold with Baas-Sullivan singularities of the type* L.

Example. Let $L = S^1$ be such that it represents $\eta \in \Omega_1^{\mathrm{spin}} = \mathbf{Z}_2$. Let $\Omega_*^{\mathrm{spin},\eta}(-)$ be the bordism theory of spin manifolds with η-singularities. There exists a Dirac operator on such spin manifold with η-singularities. It gives a natural transformation

$$\alpha^\eta : \Omega_*^{\mathrm{spin},\eta} \to ko_*^\eta = ku_*$$

which evaluates its index.

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Theorem.(B, 2001) *Let X be a simply connected spin manifold with nonempty η-singularity of dimension $n \geq 7$. Then X admits a psc-metric if and only if $\alpha^\eta([X]) = 0$ in the group $ko_n^\eta \cong ku_n$.*

Let $L_1 = \langle 2 \rangle$ be two points, $L_2 = S^1$ be as above and $L_3 = B^8$ be a Bott manifold equipped with a Ricci-flat metric. Denote:

$$\Sigma_1 = (P_1), \quad \Sigma_2 = (P_1, P_2), \quad \Sigma_3 = (P_1, P_2, P_3),$$

Let $\Omega_*^{\mathrm{spin},\Sigma_i}(-)$ be the corresponding bordism theories of manifolds with Baas-Sullivan singularities of the type Σ_i.

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$$\alpha^{\Sigma_i} : \Omega_*^{\mathrm{spin},\Sigma_i} \to ko_*^{\Sigma_i}, \quad i = 1, 2, 3, \quad \text{where} \quad ko_*^{\Sigma_1}(-) = ko_*(-; \mathbf{Z}_2),$$

$$ko_*^{\Sigma_2} = k(1)_*, \quad ko_n^{\Sigma_3} = \begin{cases} \mathbb{Z}_2 & \text{if } n = 0, 2, 4, 6 \\ 0 & \text{else} \end{cases}$$

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Theorem. (B, 2001) *Let X_Σ be a simply-connected spin manifold with non-empty Σ_i-singularities, $d_1 = 6$, $d_2 = 9$.*

- *If $i = 1, 2$ and $\dim X_\Sigma = n \geq d_i$. Then X_Σ admits a psc-metric iff $\alpha^{\Sigma_i}(X_\Sigma) = 0$ in $ko_n^{\Sigma_i}$.*
- *If $i = 3$ and $\dim X_\Sigma = n \geq 15$. Then X_Σ admits a psc-metric.*

2. Manifolds With Fibered Singularities. General setting.

Fix (L, g_L), where g_L is a metric with constant scalar curvature $s_L \geq 0$. **Assume g_L has a non-trivial isometry group $\mathrm{Iso}(g_L)$.** Fix a subgroup $G \subset \mathrm{Iso}(g_L)$.

Let X be a compact manifold with boundary ∂X which is a total space of the bundle $p : \partial X \to \beta X$ with a fiber L and a structure group G.

The group G acts on the cone $C(L)$; there is an associate bundle

$$\tilde{p} : N(\beta X) \to \beta X \quad \text{with a fiber } C(L) \text{ and structure group } G.$$

$N(\beta X)$ is a singular manifold with a boundary $\partial N(\beta X) = \partial X$ with a singular locus βX. We define X_Σ (**a manifold with (L, G)-singularities**) as

$$X_\Sigma := X \cup_{\partial M} -N(\beta X).$$

2.1. Well-adapted (or Wedge) Metrics. Let $\dim L = \ell$ be given a metric g_L with scalar curvature $s_L = \text{const} \geq 0$.

Then a conical metric $dr^2 + r^2 g_L$ on $C(L)$ has the scalar curvature

$$(s_L - s_\ell) r^{-2}, \quad \text{where} \quad s_\ell = \ell(\ell - 1).$$

We normalize g_L: $s_L = s_\ell$, thus the cone $C(L)$ is scalar flat.
A metric g on $X_\Sigma = X \cup_{\partial M} -N(\beta X)$ is **well-adapted** or **well-adapted wedge** metric if

- $g = dr^2 + r^2 g_L + \tilde{p}^* g_{\beta X}$ on $N(\beta X)$;
- $g = g_{\partial X} + dt^2$ near ∂X.

Remark: If g is psc-metric on X_Σ, then $g_{\beta X}$ is psc-metric.

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Question: Does there exist a well-adapted psc-metric on X_Σ?

Remark. There is a set of interesting examples, once we assume that X and βX are both **spin manifolds.**

2.3. Spin^c-manifolds. Let $(M, \mathcal{L})$ be **non-spin** spin^c-manifold; $\mathcal{L} \to M$ is a complex line bundle with $c_1(\mathcal{L}) \equiv w_2(M) \bmod 2$.

Let $c : M \to \mathbf{CP}^k \subset \mathbf{CP}^\infty$ be a classifying map for $\mathcal{L} \to M$. Let $B = c^{-1}(\mathbf{CP}^{k-1})$:

$$\begin{array}{ccc} M & \xrightarrow{\ c\ } & \mathbf{CP}^k \\ \Big\uparrow \text{inclusion} & & \Big\uparrow \text{inclusion} \\ B & \xrightarrow{\ c|_B\ } & \mathbf{CP}^{k-1} \end{array}$$

In particular, $\mathcal{L}|_B \to B$ is the normal bundle of the inclusion $B \subset M$. The submanifold B is **dual to $\mathcal{L} \to M$**. We identify M with X_Σ:

$$\beta X := B, \quad N(\beta X) := N(B), \quad X := M \setminus N(B) \text{ (a closure of)}.$$

By construction, $c_1(\mathcal{L})$, and thus also $w_2(X)$, is trivial on the complement $X = M \setminus N(\beta X)$, thus X is spin.

Observation: The manifold βX is spin:

$$\begin{aligned} w_2(\beta X) + \left(c_1(\mathcal{L}) \mod 2\right) &= w_2\left(TX|_{\beta X}\right) = \iota^* w_2(X) \\ &= \left(c_1(\mathcal{L}) \mod 2\right). \end{aligned}$$

We obtain a principal S^1-bundle $\partial X \to \beta X$, and

$$X_\Sigma := M = X \cup_{\partial X} -N(\beta X). \text{ (Here } L = S^1, \quad G = S^1.)$$

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Theorem. (Bérard-Bergery, 1981) *Let Z be a compact manifold with free S^1-action. Then Z admits an S^1-invariant psc-metric iff the orbit manifold $B = Z/S^1$ admits a psc-metric.*

Example 1. Consider $K3$, choose $c \in H^2(K3; \mathbf{Z}) = \mathbf{Z}^{22}$:

$$\begin{array}{ccc} Y^5 & \to & E(S^1) \\ \Big\downarrow{\scriptstyle S^1} & & \Big\downarrow \\ K3 & \xrightarrow{c} & \mathbf{CP}^\infty \end{array} \qquad X_\Sigma := X \cup_{\partial X} -N(K3)$$

Here Y^5 is a spin manifold with $\pi_1 Y^5 = 0$, and $Y^5 = \partial X$ for some spin manifold X with $\pi_1 X = 0$.

Then X_Σ admits a psc-metric, but no such metric which is S^1-invariant on $N(K3)$.

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Example 2. We start with $\widetilde{\mathbf{CP}}^5 := \Sigma^{10} \# \mathbf{CP}^5$, where Σ^{10} is a homotopy sphere with $\alpha(\Sigma^{10}) \neq 0$. Then $\widetilde{\mathbf{CP}}^5$ is a "fake complex projective space" and it has a principal S^1-bundle $\Sigma^{11} \to \widetilde{\mathbf{CP}}^5$, where Σ^{11} is an exotic homotopy sphere. There exists a spin manifold X with $\partial X = \Sigma^{11}$. Thus $X_\Sigma := X \cup_{\partial X} -N(\widetilde{\mathbf{CP}}^5)$ does not admit a well-adapted psc-metric (for any choice of X).

More on Spin^c-manifolds. Let $(M, \mathcal{L})$ be a **non-spin spin^c-manifold,** with a decomposition $M = X \cup_{\partial X} -N(B)$, where B is a dual to $\mathcal{L}$. We choose a well-adapted metric g_M on M (not necessarily psc), a hermitian metric h on $\mathcal{L}$, and a (unitary) connection $A_{\mathcal{L}}$ on $\mathcal{L}$.

This data gives us the spin^c-Dirac operator $\not{\partial}_{(M,\mathcal{L})}$ on $(M, \mathcal{L})$. We have the Lichnerowicz-Weitzenböck formula:

$$\not{\partial}^2_{(M,\mathcal{L})} = \nabla^* \nabla + \frac{1}{4} s_{g_M} + \mathcal{R}_{A_{\mathcal{L}}}.$$

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Note: Since the restriction $\mathcal{L}|_X$ is trivial, we can choose the connection $A_{\mathcal{L}}$ to be flat on $\mathcal{L}|_X$. Then the restriction $\not\partial_X = \not\partial_{(M,\mathcal{L})}|_X$ is the usual spin Dirac operator on X.

Assuming that g_M restricts to a psc-metric $g_{\partial X}$, we obtain a proper APS-boundary problem for the Dirac operator on $(X, \partial X, g_{\partial X})$. Let $\not\partial_{(X,\partial X,g_{\partial X})}$ be the resulting Dirac operator.

Observation: Take a second look at the formula:

$$\partial\!\!\!/^2_{(M,\mathcal{L})} = \nabla^*\nabla + \frac{1}{4} s_{g_M} + \mathcal{R}_{A_{\mathcal{L}}}.$$

If the metric g_M is a well-adapted psc-metric, we can deform g_M on the fibers of $N(B) \to B$ to make it equal to that on a round hemisphere $S^2_+ \subset S^2(r)$ with the hemispherical fibers having small diameter r and thus big curvature.

That implies we can make s_{g_M} highly positive without changing the curvature term $\mathcal{R}_{A_{\mathcal{L}}}$. This allows us to bound the square of the Dirac operator $\partial\!\!\!/^2_{(M,\mathcal{L})}$ away from 0. Thus $\alpha^{\mathrm{spin}^c}(M,\mathcal{L}) = 0$, where $\alpha^{\mathrm{spin}^c} : \Omega_n^{\mathrm{spin}^c} \to KU_n$ is the index map.

Note: We obtain the following Dirac operators which are relevant for existence of a well-adapted psc-metric on $(M, \mathcal{L})$:

- The Dirac operator $\partial\!\!\!/_B$ on spin manifold B.
- The spin^c-Dirac operator $\partial\!\!\!/_{(M,\mathcal{L})}$ on M twisted by the complex line bundle $\mathcal{L} \to M$.
- The Dirac operator $\partial\!\!\!/_X$ on X with the Atiyah-Patodi boundary condition on ∂X (where the metric $g|_{\partial X}$ is determined by the metric $g|_B$ and the flat metric on S^1).

Theorem$^{\mathrm{spin}^c}$ A. (B-Rosenberg, 2020) *Let $(M, \mathcal{L})$ be a non-spin spin^c-manifold, $M = X \cup_{\partial X} -N(B)$, $\dim M = n \geq 7$, where M, B are simply-connected. Assume*

- $\alpha^{\mathrm{spin}^c}(M, \mathcal{L}) = 0$ *in KU_n,*
- $\alpha(B) = 0$ *in KO_{n-2},*
- $\alpha^{\mathrm{rel}}(X, \partial X, g_{\partial X}) = 0$ *in KO_n for some psc-metric g_B (which determines $g_{\partial X}$)*

Then $(M, \mathcal{L})$ admits a well-adapted psc-metric.

Theorem$^{\mathrm{spin}^c}$ **A.** (B-Rosenberg, 2020) *Let* $(M, \mathcal{L})$ *be a non-spin* spin^c*-manifold,* $M = X \cup_{\partial X} -N(B)$, $\dim M = n \geq 7$, *where* M, B *are simply-connected. Assume*

- $\alpha^{\mathrm{spin}^c}(M, \mathcal{L}) = 0$ *in* KU_n,
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- $\alpha^{\mathrm{rel}}(X, \partial X, g_{\partial X}) = 0$ *in* KO_n *for some psc-metric* g_B *(which determines* $g_{\partial X}$*)*

Then $(M, \mathcal{L})$ *admits a well-adapted psc-metric.*

Remark: It turns out that in order to prove **Theorem**$^{\mathrm{spin}^c}$ **A** we have to understand more about geometry of the index map

$$\alpha^{\mathrm{spin}^c} : \Omega_n^{\mathrm{spin}^c} \to KU_n.$$

2.4. Geometry of α^{spin^c}. Let $(M, \mathcal{L})$ be a non-spin spin^c-manifold as above, and g be a **usual Riemannian metric** on M. Let

$$s_g^{\mathcal{L}} := s_g + 4\mathcal{R}_{\mathcal{L}}.$$

be the **$\mathcal{L}$-twisted scalar curvature** of g. Clearly $s_g^{\mathcal{L}}$ depends on a choice of the hermitian metric h on $\mathcal{L}$ and the connection $A_{\mathcal{L}}$.

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Theorem$^{\mathrm{spin}^c}$ B.(B.-Rosenberg, 2019) *Let $(M, \mathcal{L})$ be a simply connected non spin* spin^c *manifold with $\alpha^{\mathrm{spin}^c}(M, \mathcal{L}) = 0$ in KU_n with $n = \dim M \geq 5$. Then M admits a Riemannian psc-metric g, a hermitian metric h and a connection $A_{\mathcal{L}}$ such that $s_g^{\mathcal{L}} > 0$.*

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Example. Let $M = B^8 \# \mathbf{CP}^4$, where B^8 is a Bott manifold, i.e. B is spin, $\pi_1 B^8 = 0$ and $\hat{A}(B^8) = 1$. Then M is *spin^c* and not spin. (The *spin^c*-structure is given by the line bundle $\mathcal{L} \to M$ originated from the Hopf bundle on $\mathbf{CP}^4$). Then M admits a psc-metric, however, there are no metric g on M and a connection A on $\mathcal{L} \to M$ such that $s_g^{\mathcal{L}} > 0$.

To prove **Theorem**$^{\mathrm{spin}^c}$ **B**, we use the same old idea due to Stephan Stolz: **transfer**. Again, we use our old friends

$$\mathbf{HP}^2 \quad \text{and} \quad \mathbf{CP}^2$$

and corresponding isometry groups

$$PSp(3) \quad \text{and} \quad SU(3)$$

to get the **transfer homomorphisms:**

$$T^{\mathbf{HP}^2} : \ \Omega_{n-8}^{\mathrm{spin}^c}(BPSp(3)) \to \Omega_n^{\mathrm{spin}^c}$$

$$T^{\mathbf{CP}^2} : \ \Omega_{n-4}^{\mathrm{spin}^c}(BSU(3)) \to \Omega_n^{\mathrm{spin}^c}$$

Theorem$^{\mathrm{spin}^c}$ *The kernal of the index map* $\alpha^c : \Omega_n^{\mathrm{spin}^c} \to KU_n$ *is generated by the images* $\mathrm{Im}(T^{\mathbf{HP}^2})$ *and* $\mathrm{Im}(T^{\mathbf{CP}^2})$ *for* $n \geq 8$.

3. The case when $L = G/H$ is a homogeneous space.

Theorem$^{G/K}$ A. (B.-Rosenberg-Piazza, 2020) *Let $L = G/H$ be a homogeneous space, where G is a compact semi-simple Lie group, and g_L be a G-invariant metric with $s_L = s_\ell$. Let $X_\Sigma = X \cup_{\partial X} -N(\beta X)$ be a manifold with (L, G)-singularities, where $p : \partial X \to \beta X$ comes from the principoal G-bundle. Then*

(1) *∂X admits a G-invariant psc-metric;*

(2) *βX admits a psc-metric iff $N(\beta X)$ admits a well-adapted psc-metric.*

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(1) *∂X admits a G-invariant psc-metric;*

(2) *βX admits a psc-metric iff $N(\beta X)$ admits a well-adapted psc-metric.*

Example. Let $\beta X = K3$ and $L = G = Sp(1)$. Let $S^4 \to \mathbf{HP}^\infty$ be the inclusion of 4-skeleton, it gives a principal $Sp(1)$-bundle over S^4. A degree one map $K3 \to S^4$ gives a principal $Sp(1)$-bundle $Y^7 \to K3$. Since $\Omega_7^{\mathrm{spin}} = 0$, there exists spin X with $\partial X = Y^7$. There is such X that the spin manifold $X_\Sigma = X \cup -N(\beta X)$ is simply connected and $\hat{A}(X_\Sigma) = 0$. Thus X_Σ has a psc-metric. However, X_Σ does not admit a well-adapted psc-metric.

Assume all above manifolds are spin, and $L = G/H$ as above, $\ell = \dim L$. Let $\Omega_*^{\mathrm{spin},(L,G)\text{-fb}}(-)$ be the bordism theory manifold with (L, G)-singularities. This yields an exact triangle:

$$\begin{array}{ccc} \Omega_*^{\mathrm{spin}} & \xrightarrow{\quad i \quad} & \Omega_*^{\mathrm{spin},(L,G)\text{-fb}} \\ & {}_{T}\nwarrow \quad \swarrow_{\beta} & \\ & \Omega_*^{\mathrm{spin}}(BG) & \end{array}$$

Let $X_\Sigma = X \cup -N(\beta X)$ represent a class in $\Omega_n^{\mathrm{spin},(L,G)\text{-fb}}$.

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Let $X_\Sigma = X \cup -N(\beta X)$ represent a class in $\Omega_n^{\mathrm{spin},(L,G)\text{-fb}}$.

Note: The above setting gives well-defined index $\alpha_{\mathrm{cyl}}(X) \in KO_n$

Then we have two indices:

$$\alpha_{\mathrm{cyl}}(X) \in KO_n \quad \text{and} \quad \alpha(\beta X) \in KO_{n-\ell-1}$$

of the corresponding Dirac operators on X and βX.

We have two well-defined homomorphisms:

$$\alpha_{\mathrm{cyl}} : \quad \Omega_n^{\mathrm{spin},(L,G)\text{-fb}} \to KO_n$$

$$\alpha : \quad \Omega_n^{\mathrm{spin},(L,G)\text{-fb}} \to KO_{n-\ell-1}$$

Theorem$^{G/K}$ B. (B.-Rosenberg-Piazza, 2020) *Let $X_\Sigma = X \cup -N(\beta X)$ be a closed (L,G)-singular spin manifold with* $\dim X_\Sigma = n \geq 6$. *Assume that M, βM and G are all simply connected, and suppose that $L = G/K$ is a spin boundary, say $L = \partial \bar{L}$, with the G-invarinat metric g_L on L extending to a psc G-invariant metric on $\bar{L}$.*

Assume $\alpha_{\mathrm{cyl}}(X) = 0$ in KO_n and $\alpha(\beta X) = 0$ in $KO_{n-\ell-1}$. Then X_Σ admits a well-adapted metric of positive scalar curvature.

Theorem$^{G/K}$ C. (B.-Rosenberg-Piazza, 2020)
Let $X_\Sigma = X \cup -N(\beta(X))$ be a closed (L, G)-singular spin manifold with $L = \mathbf{HP}^{2k}$ and $G = Sp(2k+1)$. Assume $\partial X = \beta X \times L$, i.e. the singularities are of Baas-Sullivan type.

Then if X and βX are both simply connected and $\dim X = n \geq 6 + 8k$, X_Σ has an adapted psc-metric iff the α-invariants $\alpha_{\mathrm{cyl}}(X) = 0$ in KO_n and $\alpha(\beta X) = 0$ in KO_{n-8k-1} both vanish.

Key observation: *The bordism class of $\mathbf{HP}^{2k}$ is not a zero-divisor in the spin bordism ring Ω_*^{spin}.*

This fails for $\mathbf{HP}^{2k+1}$, since these annihilate torsion classes in the kernel of the forgetful map $\Omega_*^{\mathrm{spin}} \to \Omega_*^{SO}$.

We expect that the case of Baas-Sullivan singularities with $L = \mathbf{HP}^{2k+1}$ should work as well.

4. Remarks: Most of the discussed results could be found in

(1) B. Botvinnik, J. Rosenberg, *Positive scalar curvature on manifolds with fibered singularities*, arXiv:1808.06007

(2) B. Botvinnik, P. Piazza and J. Rosenberg, *Positive scalar curvature on simply connected spin pseudomanifolds*, to appear in Journal of Topology and Analysis,

(3) B. Botvinnik, J. Rosenberg, *Positive scalar curvature on* **Pin**$^{\pm}$ *and* **Spin**^{c}*-manifolds*, arXiv:2103.00617

The approach we discussed could be generalized to the non-simply connected manifolds with fibered singularities. Then the secondary index invariants play a major role.

(4) B. Botvinnik, P. Piazza and J. Rosenberg, *Positive Scalar Curvature on Spin Pseudomanifolds: the Fundamental Group and Secondary Invariants*, SIGMA 17 (2021), 062, 39 pages.

5. A Few Open Problems:

(1) Let $(M, \mathcal{L})$ be a spin^c-manifold and $\mathcal{R}^{\mathcal{L},\mathrm{psc}}(M) \neq \emptyset$ be the space of pairs (g, A) such that $s_g^{\mathcal{L}} > 0$. Prove or disprove that the index-difference map

$$\textbf{indexdiff} : \mathcal{R}^{\mathcal{L},\mathrm{psc}}(M) \to \Omega^{\infty+n+1}\mathbf{KU}$$

induces a surjection $\pi_* \mathcal{R}^{\mathcal{L},\mathrm{psc}}(M) \otimes \mathbf{Q} \to KU_{*+n+1} \otimes \mathbf{Q}$.

(2) Let $X_\Sigma = X \cup -N(\beta X)$ be a spin simply connected (L, G)-singular manifold, where (L, G) be one of the types we discussed. Investigate homotopy type of the space $\mathcal{R}^{\mathrm{psc}}(X_\Sigma)$ of psc well-adapted metrics.

Remark: The spaces $\mathcal{R}^{\mathrm{psc}}(X_\Sigma)$ are homotopy-invariant under appropiate surgeries:

(5) B. Botvinnik, M. Walsh, *Homotopy Invariance of the Space of Metrics with Positive Scalar Curvature on Manifolds with Singularities*, SIGMA 17 (2021), 034, 27 pages

THANK YOU!