L^2-cohomology of quasi-fibered boundary metrics

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L^2-cohomology

Let (X, g) be a complete Riemannian manifold. There is then a complex of L^2-forms

$$\cdots \longrightarrow L^2_d\Omega^q(X,g) \xrightarrow{d_q} L^2_d\Omega^{q+1}(X,g) \xrightarrow{d_{q+1}} \cdots$$

with

$$L^2_d\Omega^q(X,g) := \{\eta \in L^2\Omega^q(X,g) \mid d\eta \in L^2\Omega^{q+1}(X,g)\}.$$

L^2-cohomology is the cohomology of this complex :

$$H^q_{(2)}(X,g) = \frac{\ker d_q}{\operatorname{Im} d_{q-1}}$$

As the name suggests, L^2-cohomology depends on the metric g, but only through its quasi-isometric class :

$$\frac{g'}{C} < g < Cg' \quad \Longrightarrow \quad H^q_{(2)}(X, g) = H^q_{(2)}(X, g').$$

In general, the L^2-cohomology groups can be infinite dimensional. When this happens, reduced L^2-cohomology,

$$\overline{H}^q_{(2)}(X, g) := \ker d_q / \overline{\operatorname{Im} d_{q-1}},$$

can still be finite dimensional. Reduced L^2-cohomology can be identified with the space of L^2-harmonic forms through Kodaira decomposition :

$$L^2\Omega^q(X, g) = L^2\mathcal{H}^q(X, g) \oplus \overline{d\Omega_c^{q-1}(X)} \oplus \overline{\delta_g \Omega_c^{q+1}(X)}.$$

To compute (reduced) L^2-cohomology, one needs to understand the behavior of the metric at infinity, for instance using a natural compactification.

If $w : X \to \mathbb{R}^+$ is a positive function, one can also consider a weighted version of L^2-cohomology :

$$\mathrm{WH}^q(X, g, w) := \frac{\{\eta \in wL^2\Omega^q(X, g) \mid d\eta = 0\}}{\{d\mu \in wL^2\Omega^q(X, g) \mid \mu \in wL^2\Omega^{q-1}(X, g)\}}.$$

In practice, some of these weighted versions are often easier to compute and can give insight about (reduced) L^2-cohomology.

Fibered Boundary metrics

Let M be a compact manifold with boundary ∂M equipped with a fibered bundle $\phi : \partial M \to S$. On $\mathbb{R}^+ \times \partial M$, consider the metric

$$dr^2 + r^2 \phi^* g_S + \kappa, \qquad (*)$$

where g_S is a metric on S and $\phi^* g_S + \kappa$ is a metric on ∂M such that ϕ is a Riemannian submersion from $(\partial M, \phi^* g_S + \kappa)$ to (S, g_S). A **fibered boundary (FB) metric** g on $M \setminus \partial M$ is a Riemannian metric asymptotic to (*) outside a compact set. The tangent cone at infinity of such a metric is

$$\frac{g}{t^2} \xrightarrow[t\to\infty]{} dr^2 + r^2 g_S \quad \text{on} \quad \mathbb{R}^+ \times S.$$

Many important classes of metrics are FB-metrics :

1. Asymptotically Euclidean (AE) metrics : $\partial M = S = \mathbb{S}^n$, g_S standard round metric ;

2. Asymptotically Locally Euclidean (ALE) metrics : $\partial M = S = \mathbb{S}^n/\Gamma$, g standard metric ;

3. Asymptotically conical (AC) metrics : $\partial M = S$, $\phi = \mathrm{Id}$;

4. Asymptocially locally flat (ALF) metrics : $\phi : \partial M \to S$ is a circle bundle above $S = \mathbb{S}^2$ or $S = \mathbb{RP}^2$.

$$\begin{array}{ccccccc} \mathrm{AE} & \subset & \mathrm{ALE} & \subset & \mathrm{AC} & \subset & \mathrm{FB} \\ & & & & & & \cup \\ & & & & & & \mathrm{ALF}\,. \end{array}$$

The reduced L^2-cohomology of FB-metrics can be computed in terms of the intersection cohomology of the stratified space $\mathcal{M}$ obtained from M by collapsing ∂M onto S.

Theorem (Hausel-Hunsicker-Mazzeo 2004)

If $s := \dim S$ *is even,*

$$\overline{H}^q_{(2)}(M \setminus \partial M, g) \cong \mathrm{IH}^q_{m-\frac{s+2}{2}-q}(\mathcal{M}), \quad m := \dim M,$$

while if s *is odd,*

$$\overline{H}^q_{(2)}(M \setminus \partial M, g) \cong \mathrm{Im}\left(\mathrm{IH}^q_{m-\frac{s+1}{2}-q}(\mathcal{M}) \to \mathrm{IH}^q_{m-\frac{s-1}{2}-q}(\mathcal{M})\right).$$

For AC-metrics, the description can be given in terms of relative and absolute cohomology groups.

Corollary (Melrose 1994, Hausel-Hunsciker-Mazzeo 2004)

If g is an AC-metric,

$$\overline{H}^q_{(2)}(M \setminus \partial M, g) \cong \begin{cases} H^q(M, \partial M), & q < \frac{m}{2}, \\ \mathrm{Im}\left(H^q(M, \partial M) \to H^q(M)\right), & q = \frac{m}{2}, \\ H^q(M), & q > \frac{m}{2}. \end{cases}$$

Let me highlight three important ingredients in the proof of the result of Hausel-Hunsicker-Mazzeo :

1. They use a **fibered cusp metric** g_{fc}, which a metric conformally related to the FB-metric g,

$$g_{fc} = \frac{g}{r^2},$$

hence asymptotic to

$$\frac{dr^2}{r^2} + \phi^* g_S + \frac{\kappa}{r^2}$$

outside a compact set ;

2. They use the L^2-**Künneth formula** of Zucker for warped products of metrics ;

3. L^2-harmonic forms have a polyhomogeneous expansion at infinity, in particular they decay faster at infinity than a general L^2-form.

Quasi-fibered boundary metrics

As the name suggests, Quasi-fibered boundary metrics are a generalization of fibered boundary metrics. The intuitive idea behind their definition is the following : allow a tangent cone at infinity

$$dr^2 + r^2 g_S$$

with (S, g_S) possibly singular. More precisely, we allow S to be a stratified space and g_S to be a wedge metric. For instance, we can take S to be an orbifold and g_S to be an orbifold metric. They include the QALE-metrics of Joyce and the QAC-metrics of Degeratu-Mazzeo :

$$\text{ALE} \subset \text{QALE} \subset \text{QAC} \subset \text{QFB}.$$

Example (Carron 2011)

On the Hilbert scheme X_n of n points on $\mathbb{C}^2$, the Nakajima metric g_n is QALE, a particular case of QFB-metrics. The tangent cone at infinity is

$$(\mathbb{C}^2)^n_0/\mathcal{S}_n \quad \text{with} \quad (\mathbb{C}^2)^n_0 := \{q = (q_1, \dots, q_n) \in (\mathbb{C}^2)^n \mid \sum_j q_j = 0\},$$

where $\mathcal{S}_n$ is the symetric groups of n elements. When $n = 2$, $X_2 = T^*\mathbb{CP}^1$, g_2 is an ALE-metric and the tangent cone at infinity is $\mathbb{C}^2/\mathbb{Z}_2$.

Example (Fritzsch-Kottke-Singer 2018)

The L^2-metric $\widetilde{g}_k$ of the universal cover $\widetilde{\mathcal{M}}^0_k$ of the reduced moduli space $\mathcal{M}^0_k$ of SU(2)-monopoles of magnetic charge k on $\mathbb{R}^3$ is QFB. When $k = 2$, g_2 is a FB-metric.

There are predictions on what should the reduced L^2-cohomology of these examples.

Conjecture (Vafa-Witten 1994)

$$\overline{H}^*_{(2)}(X_n, g_n) \cong \mathrm{Im}(H^*_c(X_n) \to H^*(X_n)).$$

Conjecture (Sen 1994)

$$\overline{H}^*_{(2)}(\widetilde{\mathcal{M}}^0_k, \widetilde{g}_k) \cong \mathrm{Im}(H^*_c(\widetilde{M}^0_k) \to H^*(\widetilde{M}^0_k)).$$

Theorem (Hitchin 2000)

Both conjectures holds outside middle degree. They also hold for X_2 and $\widetilde{M}^0_2$.

Theorem (Carron 2011)

The Vafa-Witten conjecture holds for $n = 3$.

Main results

Theorem (Kottke-R 2022)

The Vafa-Witten conjecture holds for all n.

Theorem (Kottke-R 2021)

Provided $\widetilde{g}_3$ is a QFB-metric as announced by Fritzsch-Kottke-Singer, the Sen conjecture holds for $k = 3$.

Idea of the proof

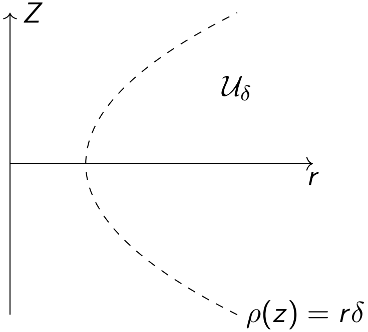
QFB-metrics admit a compactification by a stratified space with singular strata encoding the various asymptotic regimes of the metric at infinity. Near a point p on the regular part of such a strata S with link Z (a stratified space of lower depth), a local model of QFB-metric is given by

$$dr^2 + r^2\phi^* g_B + g_Z, \qquad (*)$$

where g_Z is a QFB-metric on the interior of Z and B is an open ball around p in the regular part of S.

This is a **Cartesian product** of a cone metric with a QFB-metric. However, if ρ is a positive function with $\rho \sim d_{g_Z}(\cdot, q)$ for q a point on the interior of Z, then (*) is only a model in a region of the form

$$\mathcal{U}_\delta = \{(r, b, z) \in \mathbb{R}^+ \times B \times Z \mid \rho(z) < \delta r\} \quad \text{for some } \delta > 0.$$



The model sits inside a Cartesian product, but is not one.

In particular, the L^2-Künneth formula of Zucker does not apply.

On the other hand, if r is a positive function with $r \sim d_{g_{\text{QFB}}}(\cdot, q)$ for some q, then one can consider a **Quasi-Fibered Cusp** metric

$$g_{\text{QFC}} = \frac{g_{\text{QFB}}}{r^2}$$

with local model

$$\frac{dr^2}{r^2} + \phi^* g_B + \frac{g_Z}{r^2} \quad \text{in } \mathcal{U}_\delta. \qquad (**)$$

Theorem (Kottke-R 2021)

The weighted L^2 cohomology of the modelled fibered cusp metric (**), modulo some assumptions on the weight, can be computed in terms of the weighted L^2-cohomology of the QFC-metric $\frac{g_Z}{\rho^2}$, namely

$$\mathrm{WH}^q(\mathcal{U}_\delta, g_{QFC}, w) \cong \begin{cases} \mathrm{WH}^q(Z, \frac{g_Z}{\rho^2}, w'), & q < \nu \\ \{0\}, & q \geq \nu, \end{cases}$$

for some $\nu \geq 0$ and weight w' depending on w.

Corollary (Kottke-R 2021)

For $\epsilon > 0$ small,

$$\mathrm{WH}^q(X_n, \frac{g_n}{r^2}, r^{-\epsilon}) \cong H^q_c(X_n) \quad \textit{and} \quad \mathrm{WH}^q(X_n, \frac{g_n}{r^2}, r^{\epsilon}) \cong H^q(X_n).$$

Sketch of the proof of the Vafa-Witten conjecture

By the result of Hitchin, it suffices to show that the inclusion in middle degree

$$\mathrm{Im}(H_c^{2n-2}(X_n) \to H^{2n-2}(X_n)) \hookrightarrow \overline{H}_{(2)}^{2n-2}(X_n, g_n)$$

is also **surjective**. By the previous Corollary, this follows from the following result about the decay of harmonic forms.

Theorem (Kottke-R 2022)

There exists a metric $\hat{g}_n$ quasi-isometric to g_n such that

$$L^2\mathcal{H}^{2n-2}(X_n, \hat{g}_n) \subset r^{-\epsilon} L^2\Omega^{2n-2}(X, \hat{g}_n)$$

for some $\epsilon > 0$.

To prove this theorem, we used a calculus of pseudodifferential operators adapted to *QFB*-metrics.

Thank you for your attention !