On weakly Kähler hyperbolic manifolds and geometric applications

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OUTLINE

- Review of Gromov’s theory.
- Weak versions of Kähler hyperbolicity
- Spectral theory for weakly Kähler hyperbolic manifolds
- Geometric applications

Kähler hyperbolic manifold (Gromov 1991)

Let M be a compact complex manifold and let $\pi : \tilde{M} \to M$ be its universal cover. M is said Kähler hyperbolic if there exists a Kähler form ω on M such that

$$\pi^*\omega = d\eta$$

with $\eta \in \Omega^1(\tilde{M}) \cap L^\infty\Omega^1(\tilde{M}, \tilde{\omega})$ and $\tilde{\omega} := \pi^*\omega$.

In other words

$$[\tilde{\omega}] \textit{ is trivial in } H^2_b(\tilde{M}, \tilde{\omega})$$

with $H^2_b(\tilde{M}, \tilde{\omega})$:= bounded de Rham cohomolgy of $\tilde{M}$ w.r.t $\tilde{\omega}$.

The $\partial\overline{\partial}$-lemma implies that:

- If (M, ω) is Kähler hyperbolic and μ is another Kähler form on M such that $[\omega] = [\mu]$ in $H^2_{dR}(M)$ then (M, μ) is also Kähler hyperbolic.

Examples

- Kähler manifolds homotopically equivalent to Riemannian manifolds with negative sectional curvature;
- Compact Hermitian symmetric spaces of non compact type;
- Product of Kähler hyperbolic manifolds;
- Closed complex submanifolds of Kähler hyperbolic manifolds.

Spectral/Analytic properties

Let

$$\tilde{\Delta}_{\overline{\partial},p,q} : L^2\Omega^{p,q}(\tilde{M},\tilde{\omega}) \to L^2\Omega^{p,q}(\tilde{M},\tilde{\omega})$$

be the L^2-closure of the Hodge-Kodaira Laplacian on $(\tilde{M},\tilde{\omega})$.

a) if $p+q \neq m := \dim_{\mathbb{C}}(M)$ then

$$0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},p,q}).$$

b) If $p+q = m$ then

$$\tilde{\Delta}_{\overline{\partial},p,q} : L^2\Omega^{p,q}(\tilde{M},\tilde{\omega}) \to L^2\Omega^{p,q}(\tilde{M},\tilde{\omega})$$

has closed range and 0 is an isolated eigenvalue of infinite multiplicity.

a) and b) imply that

$$\ker(\tilde{\Delta}_{\overline{\partial},p,q}) = \{0\}$$

if $p + q \neq m$ whereas $\ker(\tilde{\Delta}_{\overline{\partial},p,q})$ is infinite dimensional if $p + q = m$.

In particular $(\tilde{M}, \tilde{\omega})$ carries no L^2-holomorphic p-form for $0 \leq p \leq m - 1$ whereas the space of L^2-holomorphic m-forms is infinite dimensional.

c) $(\tilde{M}, \tilde{\omega})$ carries a linear isoperimetric inequality: there exists a positive constant c such that

$$\mathrm{vol}_{\tilde{\omega}}(\Omega) \leq c\,\mathrm{length}_{\tilde{\omega}}(\partial\Omega)$$

for any relatively compact open subset Ω with smooth boundary.

Geometric and topological properties

a) Gromov '91. Let $\mathcal{A}^p_M :=$ sheaf of holomorphic p-forms. Then

$$(-1)^{m-p}\chi(M, \mathcal{A}^p_M) > 0.$$

b) Gromov '91. The Hopf conjecture holds true for compact Kähler manifolds with negative sectional curvature:

$$(-1)^m\chi(M) > 0.$$

c) Gromov '91. M has large fundamental group: for any $x \in X \subset M$ the image

$$\mathrm{im}(i_* : \pi_1(X, x) \to \pi_1(M, x))$$

is infinite.

d) Gromov '91. $\tilde{M}$ contains no compact complex subvariety of positive dimension.

e) Gromov '91, Chen-Yang 2018. M is Kobayashi hyperbolic:

every holomorphic map

$$f : \mathbb{C} \to M$$

is constant.

f) A holomorphic line bundle over a complex manifold $L \to N$ is *big* if its Kodaira dimension is $n := \dim_{\mathbb{C}} N$. Equivalently

$$\limsup_{p \to \infty} p^{-n} \dim(H^0(N, L^p)) > 0.$$

Gromov '91. The canonical bundle K_M is big and M is projective.

The bigness of K_M follows by the fact that $\tilde{M}$ has non-trivial L^2 holomorphic forms of degree m and that $\pi_1(M)$ is large.

Weaker versions of Kähler hyperbolicity

Question: Why are they needed?

General reasons:

From the point of view of birational geometry Kähler hyperbolicity is too rigid. For instance if (M, ω) is K-H and $\pi : N \to M$ is obtained by blowing up a single point $x \in M$ then N cannot be Kähler hyperbolic because $\pi^{-1}(x) \cong \mathbb{CP}^{m-1}$ which is simply connected.

It would be desirable to have a metric condition that on one hand it implies less but still interesting properties and on the other hand it is stable e.g. through modifications.

A more concrete reason

Lang's conjecture (1986):

A smooth complex projective variety M is Kobayashi hyperbolic if and on only if any (possibly singular) subvariety $X \subseteq M$ is of general type:

if X is non-singular then K_X is big;

if X is singular there exists resolution $\pi : M \to X$ such that K_M is big.

The conjecture is largely open. The following direction of the Lang conjecture

$$M \text{ Kobayashi hyperbolic} \Longrightarrow \text{every } X \subseteq M \text{ is of general type}$$

has been verified in some cases:

(1) compact Kähler surfaces;

(2) compact Kähler manifolds with negative holomorphic sectional curvature;

(3) compact, free quotients of bounded domains.

Question: Does the Lang conjecture hold true for Kähler hyperbolic manifolds?

Let (M, ω) be Kähler hyperbolic. We have to show that each subvariety

$$X \subseteq M$$

is of general type.

We already know that M is of general type (:=K_M is big).

If $N \subset M$ is non-singular $\Rightarrow$ N is K-H $\Rightarrow$ N is of general type.

We have to deal with the case of $X \subset M$ singular.

We have to show that any resolution $\pi : N \to X \subset M$ is of general type.

Note that now $\pi^*\omega$ is only positive semidefinite on N and positive definite on $N \setminus \pi^{-1}(\mathrm{sing}(X))$.

This lead us to consider Kähler hyperbolic metrics that degenerate along a subvariety.

GOAL: Given a Kähler manifold (N, ω) that in addition carries a degenerate Kähler hyperbolic form we want to show that K_N is big.

We have to restore the spectral properties of $\tilde{\Delta}_{\overline{\partial},p,0}$ and the largeness of $\pi_1(N)$.

Let (M, ω) be a Kähler manifold. Let μ be a real, closed $(1,1)$-form such that on the universal cover $\pi : \tilde{M} \to M$ we have

$$\pi^* \mu = d\alpha$$

with $\alpha \in L^\infty\Omega^1(\tilde{M}, \tilde{\omega}) \cap \Omega^1(\tilde{M})$.

Definition

We say that (M, μ) is semi-Kähler hyperbolic if μ is positive semidefinite and positive definite on some Zariski open subset.

We say that (M, μ) quasi-Kähler hyperbolic if μ is positive semidefinite and strictly positive at some point.

Let M be a quasi-Kähler manifold and let $\mathcal{Q}_M$ be the set of quasi-Kähler forms on M.

Given $\mu \in \mathcal{Q}_M$ let $Z_{M,\mu}$ be the closed subset of points where μ is not strictly positive. Let

$$Z_M = \bigcap_{\mu \in \mathcal{Q}_M} Z_{M,\mu}.$$

Definition

A quasi-Kähler hyperbolic manifold is pseudo-Kähler hyperbolic if Z_M has zero Lebesgue measure.

Summarizing:

$$\{KH\} \subsetneq \{\text{Semi } KH\} \subseteq \{\text{Pseudo } KH\} \subseteq \{\text{Quasi } KH\}$$

Spectral theory of weakly KH manifolds I

Theorem (B.-Diverio-Eyssidieux-Trapani)

Let (M, μ) be quasi-KH. Then:

- *$(\tilde{M}, \tilde{\omega})$ carries a linear isoperimetric inequality:*

$$\mathrm{vol}_{\tilde{\omega}}(\Omega) \leq c\,\mathrm{length}_{\tilde{\omega}}(\partial\Omega)$$

  for any relatively compact open subset Ω with smooth boundary.

- $0 \notin \sigma(\tilde{\Delta}_0)$, *that is* 0 *is not in the spectrum of the Laplace-Beltrami operator* $\tilde{\Delta}_0 : L^2(\tilde{M}, \tilde{\omega}) \to L^2(\tilde{M}, \tilde{\omega})$;

- *there are no non-zero L^2 holomorphic p-forms on $\tilde{M}$ for* $0 \leq p \leq m - 1$ *w.r.t.* $\tilde{\omega}$.

Sketch of the proof

Linear Isoperimetric Inequality

(a) μ is $\tilde{d}$-bounded $\Longrightarrow \mu^m$ is $\tilde{d}$-bounded: $\tilde{\mu}^m = d(\alpha \wedge \tilde{\mu}^{m-1})$.

(b) μ is semipositive and strictly positive at some point $\Longrightarrow$

$$\int_M \mu^m > 0 \implies [\mu^m] = [\omega^m].$$

(a)+(b) $\Longrightarrow \omega^m$ is $\tilde{d}$-bounded:

$$\tilde{\omega}^m = d\eta$$

with $\eta \in L^\infty\Omega^{2m-1}(\tilde{M}, \tilde{\omega})$ and smooth.

To reach the conclusion is enough to apply Stokes theorem:

$$m!\mathrm{vol}_{\tilde{\omega}}(\Omega) = \int_\Omega \tilde{\omega}^m = \int_\Omega d\eta = \int_{\partial\Omega} \eta \leq \|\eta\|_{L^\infty(\tilde{M},\tilde{\omega})}\text{Lenght}_{\tilde{\omega}}(\partial\Omega).$$

Linear isoperimetric inequality $\Longrightarrow 0 \notin \sigma(\tilde{\Delta}_0)$ (consequence of Cheeger's inequality)

Vanishing of L^2 holomorphic p-forms, $0 < p \leq m-1$.

Let $\tilde{\mu} = d\alpha$ with $\alpha \in L^\infty\Omega^1(\tilde{M}, \tilde{\omega})$ smooth. Let A and B be open subsets of M such that $\overline{B} \subset A$ and $\mu|_A$ is strictly positive. Then there exist positive constants C and C_B such that

$$C_B \|\eta\|^2_{L^2\Omega^{p,0}(\tilde{B},\tilde{\omega})} \leq \int_{\tilde{B}} i^{p^2} \eta \wedge \overline{\eta} \wedge \tilde{\mu}^{m-p} \leq \int_{\tilde{M}} i^{p^2} \eta \wedge \overline{\eta} \wedge \tilde{\mu}^{m-p} \leq$$
$$2C \, \|\eta\|_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} \langle \Delta_{\overline{\partial},p,0}\eta, \eta \rangle^{\frac{1}{2}}_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})}$$

for each $\eta \in \mathcal{D}(\Delta_{\overline{\partial},p,0})$.

$\Longrightarrow \eta \in \ker(\tilde{\Delta}_{\overline{\partial},p,0}) \Longrightarrow \eta$ is holomorphic and $\eta|_{\tilde{B}} \equiv 0 \Longrightarrow \eta$ is identically zero.

Theorem (B.-Diverio-Eyssidieux-Trapani)

Let M be pseudo-KH and let us consider

$$\tilde{\Delta}_{\overline{\partial},p,0} : L^2\Omega^{p,0}(\tilde{M},\tilde{\omega}) \to L^2\Omega^{p,0}(\tilde{M},\tilde{\omega}).$$

Then

- $0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},p,0})$ *with* $0 \leq p \leq m-1$.
- $\tilde{\Delta}_{\overline{\partial},m,0} : L^2\Omega^{m,0}(\tilde{M},\tilde{\omega}) \to L^2\Omega^{m,0}(\tilde{M},\tilde{\omega})$ *has closed range and infinite dimensional kernel.*

  In particular $(\tilde{M},\tilde{\omega})$ *carries an infinite dimensional space of* L^2*-holomorphic m-forms.*

Sketch of the proof

We have to show that $0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},p,0})$ for $0 \leq p \leq m-1$.

We need to recall two technical lemmas:

Lemma (1)

Let M be pseudo KH and let A be an open and relatively compact subset of $M \setminus Z_M$. Then there exists $\mu \in \mathcal{Q}_M$ such that

$$\mu|_{\overline{A}}$$

is strictly positive.

Lemma (2)

Let (M, ω) be a Kähler manifold. Let $\{E(\lambda)\}_\lambda$ be the spectral resolution of

$$\tilde{\Delta}_{\overline{\partial},p,q} : L^2\Omega^{p,q}(\tilde{M}, \tilde{\omega}) \to L^2\Omega^{p,q}(\tilde{M}, \tilde{\omega}).$$

Given $\lambda_0 > 0$ there exists $\varepsilon_0(\lambda_0) > 0$ such that if U_ε is an open set of M with

$$\mathrm{vol}_\omega(U_\varepsilon) < \varepsilon$$

we have:

$$\int_{\tilde{U}_\varepsilon} |\eta|^2_{\tilde{\omega}} dvol_{\tilde{\omega}} \leq \int_{\tilde{M}\setminus\tilde{U}_\varepsilon} |\eta|^2_{\tilde{\omega}} dvol_{\tilde{\omega}}$$

for all $\eta \in \mathrm{im}(E(\lambda_0))$.

Let M be pseudo-KH. Let us fix $\lambda_0 = 1$ and let U be an open neighbourhood of Z_M such that

$$\int_{\tilde{U}} |\eta|^2_{\tilde{\omega}} \mathrm{dvol}_{\tilde{\omega}} \leq \int_{\tilde{M}\setminus\tilde{U}} |\eta|^2_{\tilde{\omega}} \mathrm{dvol}_{\tilde{\omega}}$$

for all $\eta \in \mathrm{im}(E(1))$.

Let $\mu \in \mathcal{Q}_M$ be such that $\mu|_A$ is strictly positive with A open and $\overline{M \setminus U} \subset A \subset M \setminus Z_M$.

There exist positive constants C_1 and C_2 such that

$$\|\eta\|^2_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} = \|\eta|_{\tilde{M}\setminus\tilde{U}}\|^2_{L^2\Omega^{p,0}(\tilde{M}\setminus\tilde{U},\tilde{\omega})} + \|\eta|_{\tilde{U}}\|^2_{L^2\Omega^{p,0}(\tilde{U},\tilde{\omega})} \leq$$

$$2\|\eta|_{\tilde{M}\setminus\tilde{U}}\|^2_{L^2\Omega^{p,0}(\tilde{M}\setminus\tilde{U},\tilde{\omega})} \leq C_1 \int_{\tilde{M}\setminus\tilde{U}} i^{p^2} \eta \wedge \overline{\eta} \wedge \tilde{\mu}^{m-p} \leq$$

$$C_1 \int_{\tilde{M}} i^{p^2} \eta \wedge \overline{\eta} \wedge \tilde{\mu}^{m-p} \leq C_2 \|\eta\|_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} \langle \tilde{\Delta}_{\overline{\partial},p,0}\eta, \eta\rangle^{\frac{1}{2}}_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})}.$$

Summarizing there exists a positive constant C_2 such that

$$\|\eta\|^2_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} \leq C_2\langle\tilde{\Delta}_{\overline{\partial},p,0}\eta,\eta\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})}$$

for each $\eta \in \mathrm{im}(E(1)) \subset \mathcal{D}(\tilde{\Delta}_{\overline{\partial},p,0})$.

Finally let $\gamma \in \mathcal{D}(\tilde{\Delta}_{\overline{\partial},p,0})$. We decompose γ as $\eta + \psi$ with $\eta := E(1)\gamma$. We have:

$$\langle\tilde{\Delta}_{\overline{\partial},p,0}\gamma,\gamma\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} = \langle\tilde{\Delta}_{\overline{\partial},p,0}(\eta+\psi),\eta+\psi\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} \geq$$
$$C_2\langle\eta,\eta\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} + \langle\psi,\psi\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} \geq G\langle\gamma,\gamma\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})}$$

with $G := \min\{C_2, 1\}$.

Conclusion:

$$\langle\tilde{\Delta}_{\overline{\partial},p,0}\gamma,\gamma\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})} \geq G\langle\gamma,\gamma\rangle_{L^2\Omega^{p,0}(\tilde{M},\tilde{\omega})}$$

for each $\gamma \in \mathcal{D}(\tilde{\Delta}_{\overline{\partial},p,0})$, that is

$$0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},p,0}),\ 0 \leq p \leq m-1.$$

We are left to show that $\tilde{\Delta}_{\overline{\partial},m,0}$ has closed range and infinite dimensional kernel.

Closed range: $0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},m-1,0}) \Longrightarrow 0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},m,1}) \Longrightarrow$

$\overline{\partial}_{m,0} : L^2\Omega^{m,0}(\tilde{M},\omega) \to L^2\Omega^{m,1}(\tilde{M},\omega)$ has closed range $\Longrightarrow$

$\overline{\partial}^*_{m,0} : L^2\Omega^{m,1}(\tilde{M},\omega) \to L^2\Omega^{m,0}(\tilde{M},\omega)$ has closed range

$\Longrightarrow \tilde{\Delta}_{\overline{\partial},m,0}$ has closed range.

Infinite dimensional kernel: Gromov's original strategy works also in our setting. Let

$$F := \tilde{M} \times \mathbb{C} \to \tilde{M}$$

be the trivial line bundle. Let μ be a quasi-Kähler form on M with

$$\tilde{\mu} := d\alpha$$

with $\alpha \in L^\infty\Omega^1(\tilde{M},\tilde{\omega})$ and smooth.

For each $s \in [0, 1]$ we endow F with the standard Hermitian metric and with the connection

$$\nabla_s := \nabla_0 + is\alpha$$

with ∇_0 the standard flat connection.

(a) For each $s \in [0, 1]$ there exists a (possibly non trivial) holomorphic structure on F such that ∇_s is the corresponding Chern connection.

(b) The curvature of ∇_s is

$$isd\alpha = is\tilde{\mu}.$$

Let

$$\overline{\partial}_m + \overline{\partial}_m^* : \Omega_c^{m,\bullet}(\tilde{M}) \to \Omega_c^{m,\bullet}(\tilde{M})$$

be the Dirac-Dolbeault operator associated to the Dolbeault complex

$$... \to \Omega_c^{m,q}(\tilde{M}) \stackrel{\overline{\partial}_{m,q}}{\to} \Omega_c^{m,q+1}(\tilde{M}) \to ...$$

Let

$$\overline{D}^s_m : L^2(\tilde{M}, \Lambda^{m,\bullet}(\tilde{M}) \otimes F) \to L^2(\tilde{M}, \Lambda^{m,\bullet}(\tilde{M}) \otimes F)$$

be the L^2-closure of the first order elliptic differential operator obtained by twisting $\overline{\partial}_m + \overline{\partial}^*_m$ and ∇_s.

Note: $\overline{D}^s_m$ is not Γ-equivariant, for ∇_s is not (Γ:=Deck transformation of $\tilde{M}$). $\Longrightarrow$ we cannot apply immediately Atiyah's L^2 index theorem.

However for each fixed $s \in [0, 1]$ there exists a group G_s such that

(a) G_s acts on $L^2(\tilde{M}, \Lambda^{m,\bullet}(\tilde{M}) \otimes F)$ by isometries;

(b) the action of G_s commutes with $\overline{D}^s_m$.

(a)+(b) $\Longrightarrow$ $\ker(\overline{D}^s_m)$ is invariant w.r.t. the action of G_s.

$\Longrightarrow$ $\ker(\overline{D}^s_m)$ admits a dimension with respect to G_s.

Let

$$\mathrm{ind}_{G_s}(\overline{D}_m^s) := \dim_{G_s} \ker \overline{D}_m^{s,+} - \dim_{G_s} \ker \overline{D}_m^{s,-}.$$

The following formula holds true:

$$\mathrm{ind}_{G_s}(\overline{D}_m^s) = \int_M \mathrm{Td(M)} \wedge \mathrm{ch}(K_M) \wedge \mathrm{ch}(F).$$

Note that

$$\mathrm{ch}(F) = \exp(-s\mu/2\pi) \quad \text{and} \quad \int_M \mu^m \neq 0.$$

$\Longrightarrow$ when s runs in $[0,1]$ $\mathrm{ind}_{G_s}(\overline{D}_m^s)$ is a non trivial polynomial of degree m in the variable s.

There exists $\epsilon > 0$ such that for each $s \in (0, \epsilon)$

$$\mathrm{ind}_{G_s}(\overline{D}_m^s) \neq 0$$

$\Longrightarrow$ for each $s \in (0, \epsilon)$

$$0 \in \ker(\overline{D}_m^s)$$

$\Longrightarrow$ for each $s \in (0, \epsilon)$

$$0 \in \sigma(\overline{D}_m^s)$$

$\Longrightarrow$

$$0 \in \sigma(\overline{\partial}_m + \overline{\partial}_m^*)$$

with

$$\overline{\partial}_m + \overline{\partial}_m^* : L^2\Omega^{m,\bullet}(\tilde{M}, \tilde{\omega}) \to L^2\Omega^{m,\bullet}(\tilde{M}, \tilde{\omega})$$

the Dirac-Dolbeault operator associated to the L^2-Dolbeault complex

$$\ldots \to L^2\Omega^{m,q}(\tilde{M}, \tilde{\omega}) \stackrel{\overline{\partial}_{m,q}}{\to} L^2\Omega^{m,q+1}(\tilde{M}, \tilde{\omega}) \to \ldots$$

Now we take the square of $\overline{\partial}_m + \overline{\partial}_m^*$:

$$(\overline{\partial}_m + \overline{\partial}_m^*)^2 = \bigoplus_{q=0}^{m} \tilde{\Delta}_{\overline{\partial},m,q}.$$

$0 \in \sigma(\overline{\partial}_m + \overline{\partial}_m^*) \Longrightarrow 0 \in \sigma((\overline{\partial}_m + \overline{\partial}_m^*)^2)$

$\Longrightarrow$ there exists (at least) an index q such that:

$$0 \in \sigma(\tilde{\Delta}_{\overline{\partial},m,q}).$$

On the other hand

$$0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},m,q}),\ q = 1, ..., m$$

since $\tilde{\Delta}_{\overline{\partial},m,q}$ is unitarily equivalent to $\tilde{\Delta}_{\overline{\partial},m-q,0}$ and

$$0 \notin \sigma(\tilde{\Delta}_{\overline{\partial},m-q,0}),\ q = 0, ..., m-1.$$

Only one possibility is left:

$$0 \in \sigma(\tilde{\Delta}_{\overline{\partial},m,0}).$$

Remember that $\tilde{\Delta}_{\overline{\partial},m,0}$ has closed range.

$\mathrm{ran}(\tilde{\Delta}_{\overline{\partial},m,0})$ closed **and** $0 \in \sigma(\tilde{\Delta}_{\overline{\partial},m,0}) \Longrightarrow$ 0 is an eigenvalue of $\tilde{\Delta}_{\overline{\partial},m,0}$, that is

$$\ker(\tilde{\Delta}_{\overline{\partial},m,0}) \neq 0.$$

(a) $\ker(\tilde{\Delta}_{\overline{\partial},m,0})$ is non-empty and invariant w.r.t. the action of Γ

(b) Γ is infinite

(a)+(b) $\Longrightarrow \ker(\tilde{\Delta}_{\overline{\partial},m,0})$ is infinite dimensional (in the usual sense)

$\Longrightarrow (\tilde{M}, \tilde{\omega})$ carries an infinite dimensional (in the usual sense) space of L^2 holomorphic m-forms.

Applications

Proposition

If M is quasi-KH then

$$\chi(M, \mathcal{K}_M) \geq 0.$$

If M is pseudo-KH then

$$\chi(M, \mathcal{K}_M) > 0.$$

Indeed:

$$\chi(M, \mathcal{K}_M) =$$

$$\sum_{q=0}^{m} (-1)^q \dim(H^{m,q}(M)) = \sum_{q=0}^{m} (-1)^q \dim_\Gamma(\ker(\tilde{\Delta}_{\overline{\partial},m,q})) =$$

$$\dim_\Gamma(\ker(\tilde{\Delta}_{\overline{\partial},m,q})) = \begin{cases} \geq 0 & M \text{ quasi-KH} \\ > 0 & M \text{ pseudo-KH} \end{cases}$$

Proposition

Let M be quasi-KH. There exists an analytic subvariety $Z'_M \subset M$ such that for every compact and irreducible analytic subvariety $X \subset M$ with $X \not\subseteq Z'_M$ we have

$$Im(i_* : \pi_1(X) \to \pi_1(M))$$

is infinite. In particular M has generically large fundamental group.

Z'_M is defined as

$$Z'_M := \bigcap_{\mu \in \mathcal{Q}_M} \mathrm{null}([\mu])$$

with $\mathrm{null}([\mu])$ the null locus of $[\mu]$:

$$\mathrm{null}([\mu]) = \bigcup \{V \subset M : \int_V \mu^{\dim(V)} = 0\}$$

with V compact and irreducible analytic subvarieties of M.

Theorem

Let M be a pseudo-KH manifold. Then K_M is big and M is projective.

$\pi_1(M)$ is generically large and $\tilde{M}$ carries non-trivial L^2 holomorphic m-forms;

$\Longrightarrow$ (Gromov-Kollár) K_M is *big*;

M is Kähler and carries a big line bundle $\Longrightarrow$ M is projective.

Theorem

Let (M,ω) be Kähler hyperbolic. Let $X \subseteq M$ be a (possibly singular) subvariety. Then X is of general type.

If X is non-singular this follows by Gromov '91.

If X is singular and $r : N \to X$ is a resolution of X then

$$(N, r^*\omega)$$

is semi-KH and therefore pseudo-KH.

Now the conclusion follows by B.-Diverio-Eyssidieux-Trapani.

Thanks for your attention