The Godbillon-Vey invariant in KK-theory with real coefficients

Georges Skandalis
joint work with Paolo Antonini and Sara Azzali

Université Paris-Cité
Institut de Mathématiques de Jussieu-Paris Rive Gauche

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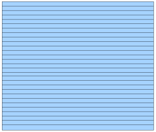
1. Foliations

1.1 Definition

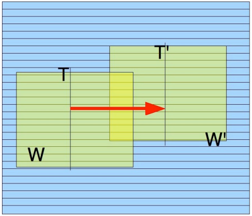
Partition into connected submanifolds local picture:



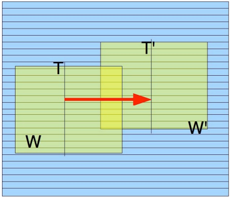
In other words: there is an open cover of M by foliation charts of the form $U \times T$ where $U \subset \mathbb{R}^p$ and $T \subset \mathbb{R}^q$.

The set T is the transverse direction and U is the leafwise or longitudinal direction.

The change of charts is of the form $f(u,t) = (g(u,t), h(t))$.



1.2 Ruelle-Sullivan current



A holonomy invariant transverse measure on a foliated manifold is a (Radon) measure μ_i on every transversal T_i of a chart $W_i = U_i \times T_i$, that agrees on the intersection of two charts.

Associated to such an invariant transverse measure (μ_i) is the Ruelle-Sullivan current:

let ω be smooth a p-form with compact support (p=dimension of the leaves); write $\omega = \sum \omega_i$ where ω_i is supported in $\Omega_i = U_i \times T_i$. Put

$$Z(\omega) = \sum_i \int_{T_i} \Big(\int_{U_i \times t} \omega_i \Big) d\mu_i(t).$$

(In this talk, we assume everything is oriented).

By invariance, does not depend on the decomposition $\omega = \sum \omega_i$.

Closed current : defines a homology class.

Connes' index theorem for measured foliations

Let $(M, \mathcal{F})$ be a compact manifold endowed with a holonomy invariant transverse measure. Let Z_{RS} be the associated Ruelle-Sullivan current.

The holonomy invariant transverse measure defines also a trace τ (positive - densely defined) on the foliation C^*-algebra and therefore a morphism $\tau : K_0(C^*(M, \mathcal{F})) \to \mathbb{R}$.

Connes' index theorem for measured foliations

Let D a longitudinally elliptic longitudinal (pseudo)-differential operator D. We have

$$\tau(\mathrm{ind}(D)) = \langle ch([\sigma(D)]) \wedge Td(F) \,|\, [Z_{RS}] \rangle.$$

1.3 The Godbillon-Vey invariant
First definition

Let M be a smooth manifold and let $\mathcal{F}$ be a foliation of codimension 1. (The leaves have dimension $n-1$ if $\dim M = n$.)

We assume that the foliation is transversaly oriented. This means, that the transverse bundle $TM/T\mathcal{F}$ - of dimension one is oriented and therefore trivial.

There exists a 1-form α on M such that $\ker(\alpha) = T\mathcal{F}$.

As $\mathcal{F}$ is a foliation, there exists a 1-form η such that $d\alpha = \alpha \wedge \eta$.

As $d(d\alpha) = 0$, we find $d\alpha \wedge \eta - \alpha \wedge d\eta = 0$, whence $\alpha \wedge d\eta = 0$, and there exists β with $d\eta = \alpha \wedge \beta$. Therefore $d\eta \wedge d\eta = 0$.

The form $\eta \wedge d\eta$ is closed. It defines a cohomology class: the Godbillon-Vey class.

One checks that this class does not depend on the choices (α, η).

The Godbillon-Vey invariant. Equivalent definition

The space of jets. Let T be an oriented manifold of dimension k.
The space of positive r-jets on T is the set $J^r T$ of smooth maps $q : \mathbb{R}^k \to T$ such that $(dq)_0$ is invertible and preserves orientation, divided by the equivalence relation $q_1 \sim_r q_2$ if q_1 and q_2 have the same Taylor expansion at 0 up to order r.

Of course an orientation preserving diffeomorphism acts on positive jets.

Let $(M, \mathcal{F})$ be a (transversally oriented) foliation, given by foliation charts $(U_i \times T_i)_{i \in I}$, we obtain a foliated manifold $J^r_{transv}(M, \mathcal{F})$ by gluing $U_i \times J^r T_i$.

The Godbillon-Vey invariant via jets

Let $(M, \mathcal{F})$ be a transversaly oriented foliation of codimension 1, with foliation charts $(U_i \times T_i)_{i \in I}$ where T_i is an open interval in $\mathbb{R}$.

$J^2 T_i = T_i \times \mathbb{R}_+^* \times \mathbb{R}$: write $q(t) = u + tx + y\dfrac{t^2}{2} + o(t^2)$.

If $h : T_i \to T_j$ is a (partial) diffeomorphism, we may write

$$(J^2 h)(q)(t) = h(u) + h'(u)(tx + y\frac{t^2}{2}) + \frac{h''(u)}{2}t^2x^2 + o(t^2).$$

So $J^2 h(u, x, y) = \big(h(u), h'(u)x, h'(u)y + h''(u)x^2\big)$.

The Jacobian matrix of $J^2 h$ is $\begin{pmatrix} h'(u) & 0 & 0 \\ h''(u)x & h'(u) & 0 \\ h''(u)y + h'''(u)x^2 & 2h''(u)x & h'(u) \end{pmatrix}$.

The 3-form $x^{-3} du \wedge dx \wedge dy$ on $J^2 T_i$ is invariant by $J^2 h$ and thus defines a 3-form on $J^2_{trans} M$.

It defines the Godbillon-Vey class and a transverse measure on $J^2_{transv}(M, \mathcal{F})$ invariant by holonomy.

2. KK-theory with real coefficients
2.1 Definitions and properties

With P. Antonini and S. Azzali, we defined $KK_{\mathbb{R}}(A,B) = \varinjlim_{D,\tau} KK(A, B \otimes D)$ where D is a (separable) C^*-algebra and τ is a trace on D.

This KK-theory is endowed with a natural Kasparov product
$KK_{\mathbb{R}}(A_1, A_2) \times KK_{\mathbb{R}}(A_2, A_3) \to KK_{\mathbb{R}}(A_1, A_3)$

In particular, an element of $KK_{\mathbb{R}}(A, \mathbb{C})$, defines a morphism $K_0(A) \to \mathbb{R}$.

2.2 $KK_{\mathbb{R}}$-element associated with an infinite trace

Given a densely defined positive trace τ on a (nonunital) C^*-algebra D, we may embed D in a trace preserving way into a C^*-algebra D_1 with a projection p of trace 1 such that $D_1 p D_1$ is dense in D_1 (using an amalgamated free product construction). Then pD_1p is Morita equivalent to D_1.

The inclusion $D \subset D_1$ defines an element in $KK(D, D_1) = KK(D, pD_1p)$ whence an element $[\tau] \in KK_{\mathbb{R}}(D, \mathbb{C})$.

2.3 $KK_{\mathbb{R}}$-element $[GV]$

The Godbillon-Vey form defines a Ruelle-Sullivan current Z_{GV} on $J^2_{transv}(M, \mathcal{F})$ and a trace τ_{GV} on the foliation algebra $C^*(J^2_{transv}(M, \mathcal{F}))$.

Now the group of 2-jets of orientation preserving diffeomorphisms of $\mathbb{R}$ mapping 0 to 0 acts freely and properly on $J^2_{transv}(M)$, preserving the foliation, with quotient M.

This group is isomorphic to the $ax + b$ group.

We obtain a Morita equivalence

$$C^*(M, F) \sim_{Morita} C^*(J^2_{transv}(M, \mathcal{F})) \rtimes (ax + b).$$

Connes' Thom isomorphism: invertible element $\beta \in KK\big(C^*(M, F), C^*(J^2_{transv}(M, \mathcal{F}))\big)$.

And thus $[GV] = \beta \otimes [\tau_{GV}] \in KK_{\mathbb{R}}(C^*(M, \mathcal{F}), \mathbb{C})$.

3. Pairing $[GV]$ with the index

Is [GV] Godbillon-Vey?
Yes !

Proposition

Let D a longitudinally elliptic longitudinal (pseudo)-differential operator D. We have $\mathrm{ind}(D) \otimes [GV] = \langle ch([\sigma(D)]) \wedge Td(F) \,|\, [Z_{GV}] \rangle$.

We may apply Connes theorem for measured foliations to $J^2_{transv}(M, \mathcal{F}))$ and the symbol $\sigma_D \otimes \mathrm{Bott}$ (writning $J^2_{transv}(M) = M \otimes \mathbb{R}^*_+ \times \mathbb{R}$).

Note that the manifold $J^2_{transv} M = M \times \mathbb{R}^*_+ \times \mathbb{R}$ is not compact... Can be done, but...

Longitudinal index theorem ⇒ measured index theorem

Recall

Longitudinal index theorem (Connes-S 91)

(M,F) foliated manifold. $\mathrm{ind}_{an} = \mathrm{ind}_{top}$

- $\mathrm{ind}_{an} \in KK(C_0(F^*), C^*(M,F))$: to a symbol $F \in K^0(F^*)$ associates the index of the corresponding longitudinal pdo in $K_0(C^*(M,F))$.
- ind_{top} is obtained using a smooth map $f : M \to \mathbb{R}^n$ such that df is injective when restricted to the leaf.
  Let $N_x = (df_x)^\perp$ the normal to the leaf.
  The map $(x,\xi) \mapsto (x, f(x)+\xi)$ with $x \in M$ and $\xi \in N_x$, $\|\xi\| < \varepsilon$, identifies the open ball N_ε in N with radius ε with an open transversal of the foliation $(M \times \mathbb{R}^n, F \times \{0\})$.

$$\boxed{\mathrm{ind}_{top} = \mathrm{Thom} \otimes i \otimes \mathrm{Bott}^{-1}} \quad \text{where}$$

  - $\mathrm{Thom} \in KK(C_0(F^*), C_0(N))$ (since $N + F = \mathbb{R}^n$);
  - i corresponds to an injection $i : C_0(N) \to C^*(M \times \mathbb{R}^n, F \times \{0\})$;
  - $\mathrm{Bott}^{-1} \in KK(C^*(M \times \mathbb{R}^n, F \times \{0\}), C^*(M,F))$.

Longitudinal index theorem $\Rightarrow$ measured index theorem

Need to compute the pairing of ind_{top} with a trace τ corresponding to the Ruelle-Sullivan cycle.
N a transversal in $(M \times \mathbb{R}^n; F \times \{0\})$. $C_0(N) \subset C^*(M, F) \otimes C_0(\mathbb{R}^n)$.
Pairing of $K^0(N)$ with $[\tau] \otimes \mathrm{Bott}^{-1}$ (n even).

Proposition

For $x \in K^0(N)$, we have $([\tau] \otimes \mathrm{Bott}^{-1})(x) = \int_N \mathrm{Chern}(x) \wedge Z_{RS}$.

Remark. The Todd class in Connes' formula comes from the Thom isomorphism $K^0(F^*) \simeq K^0(N)$.

In cohomology

More generally, let X, Y be smooth manifolds and let $f : Y \to X \times M/F$ be an etale map: Y sits in $X \times M/F$ as an open transversal.
We obtain a morphism (a bimodule) $f_* : C_0(Y) \to C_0(X) \otimes C^*(M, F)$, and therefore an ellement $[f_\tau] \in KK_{\mathbb{R}}(C_0(X), C_0(Y))$.

Note that f defines a submersion $Y \to X$ whose fibers are transversals of (M, F).

Proposition

The map

$$H^*(Y;\mathbb{R}) \overset{Chern}{\simeq} KK_{\mathbb{R}}(\mathbb{C}, C_0(Y)) \to H^*(X;\mathbb{R}) \overset{Chern}{\simeq} KK_{\mathbb{R}}(\mathbb{C}, C_0(X))$$

associated with f_τ *is* $\omega \mapsto f_!(\omega \otimes Z_{RS})$

i.e. this map is obtained by integrating ω along the fibers of $Y \to X$ using the invariant measure.

In other words, given a closed current Z on X, we have, for $y \in K_0(C_0(Y))$, $\boxed{\langle Chern(y \otimes [f_\tau])|Z\rangle = \langle Chern(y)|(Z_{RS} \wedge f^*(Z)\rangle.}$

Using cyclic cocycles

Let X be a smooth manifold, (A, τ) a C^*-algebra with a (densely defined, semi-continuous) trace.

An idempotent in $e \in M_n\big(\widetilde{C_c^\infty(X; A)}\big)$ defines $[e] \in KK_{\mathbb{R}}(\mathbb{C}, C_0(X))$.

Proposition

If Z is a closed current on X of dimension $2k$, we have $\langle \mathrm{Chern}([\mathrm{e}]|Z\rangle = \vartheta_{Z,\tau}(e, \dots, e)$ *where*

$$\vartheta_{Z,\tau}(f_0, \dots, f_{2k}) = C_k \int_Z (\mathrm{id} \otimes \tau)(f_0\, df_1\, \dots\, df_{2k}).$$

Here $f_0\, df_1\, \dots\, df_{2k} \in \Omega_c^{2k}(X) \otimes A$; C_k universal constant $C_k = \dfrac{1}{(2i\pi)^k\, k!}$

It is shown by Connes that:

- $\vartheta_{\tau,Z}$ is a cyclic $2k$-cocycle on $C_c^\infty(Y, Dom(\tau))$;
- therefore, this formula gives indeed a morphism $K_0(C_0(X, A)) \to \mathbb{C}$;
- equality holds for $A = \mathbb{C}$.

Write $KK_{\mathbb{R}}(\mathbb{C}, C_0(X)) = K_0(C_0(X)) \otimes \mathbb{R}$ to establish proposition!

What is an etale map $Y \to X \times M/F$?

Etale map $f : Y \to X \times M/F$:

- Smooth submersion $g : Y \to X$;
- open cover (Y_i) of Y;
- smooth maps $f_i : Y_i \to T_i$ open transversal of (M, F),
- smooth maps $f_{i,j} : Y_i \cap Y_j \to X \times Hol(M, F)$ (holonomy groupoid);

such that

- $(g, f_i) : Y_i \to X \times T_i$ is an open inclusion;
- $r \circ f_{i,j} = f_i,\ s \circ f_{i,j} = f_j$ and,
- for $y \in Y_i \cap Y_j \cap Y_k$, $f_{i,k}(y) = f_{i,j}(y) f_{j,k}(y)$ (cocycle property).

Given an etale map $Y \to X \times M/F$...

Consider the groupoid $\Gamma = \sqcup_{i,j} Y_i \cap Y_j \rightrightarrows \sqcup_i Y_i$

The groupoid Γ embeds as an open subgroupoid of $Y \times I \times I$ and also of $X \times G$, where

$G = \sqcup_{i,j} Hol(M,F)_{T_j}^{T_i} \rightrightarrows \sqcup_i T_i$; by $(y,i,j) \mapsto \big(g(y), f_{i,j}(y)\big)$.

Thus morphisms $C_c^\infty(\Gamma) \to C_c^\infty(Y;\mathcal{K})$ and $C_c^\infty(\Gamma) \to C_c^\infty(X; C^*(G))$.

Given a closed current Z on X, we obtain a closed current $Z_{RS} = \omega_{RS} \wedge g^*(Z)$ on Y and cyclic cocycles $\vartheta_{Z_{RS},Tr}$ on $C_c^\infty(Y;\mathcal{C}^1)$ and $\vartheta_{Z,\tau}$ on $C_c^\infty(X; C^*(G))$.

The restrictions of these two cyclic cocycles on $C_c^\infty(\Gamma)$ coincide - the result follows.

Thank you for your attention!
and happy birthday Paolo!