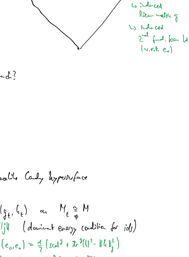
Enlargeability obstruction for spacetimes with both big bang and big crunch

1. Big bang and big crunch

$(\bar{M}, \bar{g})$ time-oriented Lorentzian manifold, globally hyperbolic

Thm (Hawking's singularity theorem)
Let $M \subsetneq \bar{M}$ be a smooth spacelike Cauchy hypersurface.
Assume:
- $(\bar{M}, \bar{g})$ satisfies the dominant energy condition (ndec): $E(v,V) \geq 0 \quad \forall V, v$ future-causal
- $\pm \operatorname{tr}_M^{\bar{g}}(\mathrm{II}) \geq n\lambda > 0$

Then: Every future-inextendible causal geodesic γ starting at M leaves the causal future of M after a finite affine parameter t.
"big bang"
"big crunch"



Guiding question: Can spacetimes have both big bang and big crunch?
→ Tackle this in terms of initial data sets

2. Initial data formulation

globally hyperbolic $\overset{\text{Bernal-Sánchez}}{\Leftrightarrow}$ $\bar{M} \cong \mathbb{R} \times M$ with $M_t = \{t\} \times M$ spacelike Cauchy hypersurface

Lorentzian metric $\bar{g}$ ⟷ initial data set (ids) (g_t, k_t) on $M_t \cong M$
$\bar{g}$ satisfies dec ⟷ (g_t, k_t) satisfies $\mu \geq |j|_g$ (dominant energy condition for ids)
where $\mu := \frac{1}{2}(\mathrm{Scal}^g + (\operatorname{tr}_g k)^2 - |k|_g^2)$
$j := \operatorname{div}_g(k - (\operatorname{tr}_g k)g)$
$\bar{g}$ satisfies strict dec ⟷ (g_t, k_t) satisfies $\mu > |j|_g$ (strict dec)
↳ dec because $E(v,V) = V^i \mu / 4$ for V lightlike or zero

From now on: M compact.

Def $\quad \mathcal{I}(M) := \left\{ (g,k) \;\middle|\; \begin{array}{l} g \text{ metric on } M \\ k \text{ sym. 2-tensor on } M \end{array} \right\}$ $\quad C^\infty$-topology
$\mathcal{I}^+(M) := \{(g,k) \in \mathcal{I}(M) \mid \mu > |j|_g\}$

Def $\quad$ We call $(g,k) \in \mathcal{I}^+(M)$ a big bang ids / big crunch ids if $\operatorname{tr}_g k \geq 0$.

Us-formulation of the question: Is there a continuous path $[-1,1] \to \mathcal{I}^+(M)$ from big bang ids to big crunch ids?
(A spacetime with Cauchy hypersurface diffeo to M, subject to strict dec and with both big bang and big crunch gives rise to such a path.)

Rmk $\quad$ All big bang ids belong to the same path-component of $\mathcal{I}^+(M)$, similarly for big crunch ids
$C^+ :=$ path-comp. of $\mathcal{I}^+(M)$ containing all big bang ids
$C^- :=$ ⟷ ⟷ big crunch ids

Question (gdc): Is $C^+ = C^-$?

Lemma $\quad$ If M carries a positive scalar curvature ($=\mu g$) metric, then $C^+ = C^- = \mathcal{I}^+(M)$.
Pf: Consider $[-1,1] \to \mathcal{I}(M)$ for a pc metric g.
$t \mapsto (g, tg)$
- Consider big bang ids / big crunch ids
- $\mu = \frac{1}{2}(\mathrm{Scal}^g + n(n-1)) \geq \frac{1}{2} n(n-1) > 0$ and $j = 0$ ⇒ path in $\mathcal{I}^+(M)$ □

Q: What about the converse?

Thm (G. preprint '19) $\quad$ M compact, spin, $\dim(M) = n \geq 2$.
If $\exists\, g \in \mathcal{M}(M) \in \mathcal{KO}^{-n}(\{pt\})$, then $C^+ \neq C^-$.
⇒ M long ago carry psc ⟸ for $n \geq 2$

Thm (G. preprint '21) $\quad$ M compact, $\dim(M) = n \geq 2$.
If M is spin and enlargeable (more generally: $\hat{A}$-area enlargeable), then $C^+ \neq C^-$.
⇒ M does not carry psc

Examples:
- Tori are enlargeable
- Non geometric 3-mfds: If M admits $K \leq 0$, then $M \# \bar{M}$ is enlargeable
- M, N enlargeable ⇒ $M \times N$ enlargeable
- N enlargeable, N total space of fiber bundle with $M \# \bar{M}$ enlargeable ⇒ $M \# \bar{M}$ enlargeable
- M orientable cover of M: M enlargeable ⟺ $M = M_1 \# \ldots \# M_k$ with $M_i = S^n$ or $M_i = T^n$ ⟺ M does not carry psc

3. Enlargeability and index theory

Proof of "spin + enlargeable ⇒ $C^+ \neq C^-$" in the case $2 \leq \dim(M)$ and M orientably enlargeable:

1st construction:

Thm (Gromov–Lawson '83) $\quad$ Let M be a compact Riemannian manifold.
Assume M is orientably enlargeable and $2 \leq \dim(M)$. Then there exist
- $(M_i \to M)_{i \in \mathbb{N}}$ finite oriented Riemannian coverings
- $(E_i \to M_i)_{i \in \mathbb{N}}$ complex vector bundles with Hermitian metric and metric connection

such that
- $\forall i \in \mathbb{N}: \quad 0 \neq \hat{A}(M_i, E_i) := \int_{M_i} \hat{A}(TM_i) \cdot \mathrm{ch}(E)$
- $\|R^{E_i}\| \longrightarrow 0$ for $i \to \infty$.

2nd construction: Given a spin manifold M, a Hermitian vector bundle (E, h) and connection ∇^E on E with curvature R^E:

- $\overline{\Sigma}M := \Sigma M \otimes \overline{\Sigma M} \longrightarrow M$ — hyperbolic spinor bundle
  - classical complex spinor bundle
  - ↳ connection ∇
  - ↳ Hermitian metric
  - ↳ (left) Clifford multiplication by TM
  - ↳ grading ι
  - for duality / the bundle: ↳ involution e_0
  - ↳ (right) $\mathcal{O}_E$-action ϵ_0
  - "capable"
- $\overline{\Sigma}^E M := \overline{\Sigma}M \otimes_E E \longrightarrow M$ — twisted hyperbolic spinor bundle
  - ↳ structures as above
- $\overline{D}^E := \underline{D}^E - \frac{1}{2}\operatorname{tr}^g(\mathrm{II}) \cdot e_0 \cdot \mathrm{id}_E$ — twisted Dirac-like operator
  - twisted Dirac operator acting on $\Gamma(\overline{\Sigma}^E M)$
  - ↳ elliptic
  - ↳ formally self-adjoint
  - ↳ odd
  - ↳ $\mathcal{O}_E$-linear
  - valuable
  - ↳ not self-adj. odd, $\mathcal{O}_E$-linear
  - Fredholm operator on $L^2(\overline{\Sigma}^E M)$

Lemma $\quad$ $\overline{D}^E$ satisfies the Schrödinger–Lichnerowicz type formula
$$(\overline{D}^E)^2 = \overline{\nabla}^* \overline{\nabla} + \frac{1}{2}\left(\mathrm{Scal}^g - \mu^E \cdot e_0\right) + \mathcal{R}^E$$
$\overline{\nabla}^* \overline{\nabla} \geq 0$, $\mathcal{R}^E(\psi \otimes e) = \sum_{ij} e_i e_j \cdot \psi \otimes R^E(e_i, e_j)e$
In particular: If $\frac{1}{2}\left(\mathrm{Scal}^g - |j|_g\right) > \|R^E\|$, then $\overline{D}^E$ is invertible.
(e.g.: $(g,k) \in \mathcal{I}^{>0}(M)$)

Pf: Let $(g_t, k_t)_{t \in [-1,1]}$ be a path with $(g_{\pm 1}, k_{\pm 1}) = (g, \pm k) \in \mathcal{I}^{>0}(M)$.
Then the path of twisted Dirac-like operators $t \mapsto \overline{D}^E_{(g_t, k_t)}$ gives rise to an element $\left[\overline{D}^E\right] \in K^{-1}(\mathbb{R}, \mathbb{R}\setminus\{0\}) \cong \mathbb{Z}$
with the properties:
- $\left[\overline{D}^E\right]$ is additive under concatenation of paths
- $\left[\overline{D}^E\right]$ depends only on the endpoints
- $\left[\overline{D}^E\right] = 0$ if the path lies within $\mathcal{I}^{>0}(M)$

$\mathcal{I}(M)$ is convex (or at least we can homotopy out any π_0), & topology invariance of the index

↳ this can be the 1st fact we know that $\left[\overline{D}^E\right] \neq 0$ in the image of $\mathrm{Dir} \in K^{-1}(T, \partial T) \to K^{-1}(T, \mathbb{R})$

Def $\quad$ For $(g_{+-}, k_{+-}), (g_{+-}, l_{+-}) \in \mathcal{I}^{>0}(M)$:
$$\text{ind-diff}^E\left((g_{+-}, k_{+-}), (g_{+-}, l_{+-})\right) := \left[\overline{D}^E\right] \in K^{-1}(\mathbb{R}, \mathbb{R}\setminus\{0\}) \cong \mathbb{Z}$$
twisted index difference
(across same path $(g_{+-}, \cdot)_{+-} \mapsto (g_{+-}, \cdot)$)

In general hard to compute, but:
Thm ("Index theorem")
Let γ be a ric. orbit in M and $\lambda > 0$ so large that $(g, \pm\lambda g) \in \mathcal{I}^{>0}(M)$. Then:
$$\text{ind-diff}^E\left((g, -\lambda g), (g, \lambda g)\right) = \hat{A}(M, E)$$

Sketch:
$$\hat{A}(M,E) \overset{\text{def}}{=} \mathrm{ind}\left(\underline{D}^E : L^2(\Sigma M \otimes E) \to L^2(\Sigma M \otimes E)\right) \in \mathcal{KO}^0(\{pt\}) \cong \mathbb{Z}$$
$$\cong \sum_i L^2(\Sigma M \otimes E)_i \;\; \underline{D}^E \quad \text{(} p_1 : \mathbb{R} \to \{pt\} \text{)}$$
$$\left[\overline{D}^E\right] \in \prod_i L^2(\Sigma M \otimes E) \otimes \mathbb{C}, \; \iota, -\epsilon_0 \overline{D}^E + t e_0 \big)$$
path $\mathbb{R} \to \overline{\Sigma}^E(M)$ $(t \mapsto (g, \pm\lambda g))$
$\cong$ $\mathbb{Z}$ $\quad$ $\cong$ $K^{-1}(\mathbb{R}, \mathbb{R}\setminus\{0\})$
$\left[\overline{D}^E\right] \in K^{-1}(\mathbb{R}, \mathbb{R}\setminus\{0\})$
path $t \mapsto \overline{D}^E_{(g, t\lambda g)}$

□

Putting the argument together:
- Assume for contradiction $C^+ = C^-$ $\Rightarrow \exists\, (g,k): I \to \mathcal{I}^+(M)$ with $(g_{\pm 1}, k_{\pm 1}) = (g, \pm k)$
- Since $I = [-1,1]$ is compact, $\mu - |j|_g > \varepsilon$ along I for some $\varepsilon > 0$.
- Choose (following Gromov–Lawson) a covering $M' \to M$ and a bundle $E' \to M'$ with $\|R^{E'}\| < \frac{\varepsilon}{2}$ and $\hat{A}(M', E') \neq 0$.
  (exists as $\|R^{E'}\| \to 0$)
- Then $(g, k, \iota): I \to \mathcal{I}^{>0}(M')$ and
$$0 \overset{\text{index-diff}^{E'}((g,-k),(g,k)) = \hat{A}(M',E') \neq 0}{\neq} \text{ind-diff}^{E'}\left((g,-k),(g,k)\right) = \hat{A}(M', E') \neq 0$$
Index theorem

↯

□