Dynamical residues of spectral ζ-functions

jointly with Nguyen Viet Dang (Jussieu)

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Introduction

Spectral zeta functions

(M, g) compact Riemannian $\implies \triangle_g$ has discrete spectrum.

Recall Riemann zeta $\zeta(\alpha) = \sum_{\lambda=1}^{\infty} \lambda^{-\alpha}$, then spectral zeta of $\triangle_g$:

$$\mathbb{C} \ni \alpha \mapsto \zeta_g(\alpha) = \sum_{\lambda \in \mathrm{sp}(\triangle_g) \setminus \{0\}} \lambda^{-\alpha}.$$

Theorem (Minakshisundaram–Pleijel, Seeley)

The function $\zeta_g(\alpha) = \mathrm{Tr}_{L^2}\left(\triangle_g^{-\alpha}\right)$ *is holomorphic on* $\mathrm{Re}\,\alpha > \frac{n}{2}$, *with meromorphic continuation to* $\alpha \in \mathbb{C}$ *and poles at* $\{\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1\}$.

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+local version with densities:

$\alpha \mapsto \triangle^{-\alpha}(x,x)$ holomorphic on $\mathrm{Re}\,\alpha > \frac{n}{2}$, with meromorphic continuation to $\alpha \in \mathbb{C}$ and poles at $\{\frac{n}{2}, \frac{n}{2}-1, \dots, 1\}$, smooth in $x \in M$.

Spectral action for Euclidean gravity

The heat kernel expansion (small t expansion of $e^{-t\triangle_g}(x,x)$) relates $\triangle_g$ with invariants, in particular scalar curvature $R_g(x)$.

Theorem (elliptic theory + Connes, Kalau–Walze, Kastler)

When dim$(M) = n \geqslant 4$*,*

$$\operatorname*{res}_{\alpha=\frac{n}{2}-1} \mathrm{Tr}_{L^2}\left(\triangle_g^{-\alpha}\right) = \frac{\int_M R_g(x)}{6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)}.$$

Local version for diagonal value $x = y$ of Schwartz kernel $\triangle_g^{-\alpha}(x,y)$:

$$\operatorname*{res}_{\alpha=\frac{n}{2}-1} \triangle_g^{-\alpha}(x,x) = \frac{R_g(x)}{6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)}.$$

— This is a spectral action for Euclidean gravity: $\delta_g R_g = 0$ is equivalent to Euclidean analogue of Einstein equations.

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— One can replace $\triangle_g$ by D^2. Motivates definitions of curvature in non-commutative geometry Connes–Marcolli '08, Connes–Moscovici '14

— Poles are geometric $\Rightarrow$ locality of counterterms in zeta function regularisation in QFT Hawking '77

Guillemin–Wodzicki residue

For $A \in \Psi^m_{\text{cl}}(M)$ with symbol $a \sim \sum_j a_{m-j}$, Guillemin–Wodzicki residue density:

$$\text{res}\, A := \frac{1}{(2\pi)^n} \left(\int_{\mathbb{S}^{n-1}} a_{-n}(x;\xi) \iota_V d^n\xi \right) d^n x$$

Then, $\quad \text{res}_{\alpha=\frac{n}{2}-1} \triangle_g^{-\alpha}(x,x) = \frac{1}{2}\big(\text{res}\, \triangle_g^{-\frac{n}{2}+1}\big)(x).$

- Equality with a Dixmier trace Connes '88.
- Identity generalizes to complex powers of elliptic ΨDOs.
- Notion of Guillemin–Wodzicki residue generalizes to FIOs.

Two related questions:

Is it possible to define res A without ΨDO or FIO calculus?

Can we replace $\triangle_g$ by non-elliptic, non-positive $\Box_g$ and work in Lorentzian signature?

I. A dynamical definition of residue

Scaling towards the diagonal

Let $\Delta = \{(x,x) \,|\, x \in M\}$.

A vector field X is radial (or Euler) if $Xf = f$ modulo quadratically vanishing terms for all f with $f|_\Delta = 0$.

Locally there are coordinates $(x^i, h^i)_{i=1}^n$ s.t. $\Delta = \{h^i = 0\}$ and $X = \sum_{i=1}^n h^i \partial_{h^i}$.

$u \in \mathcal{D}'_\Gamma(\mathcal{U})$ is log-polyhomogeneous if

$$e^{-tX} u = \sum_{p \leqslant k \leqslant N, 0 \leqslant i \leqslant l-1} e^{-tk} \frac{(-1)^i t^i}{i!} (X-k)^i u_k + \mathcal{O}_{\mathcal{D}'_\Gamma(\mathcal{U})}(e^{-t(N+1-\varepsilon)}).$$

Pollicott–Ruelle resonances of the flow e^{-tX} are the poles of

$$\int_0^\infty e^{-tz} \langle (e^{-tX} u), \varphi \rangle \, dt = \sum_{k=p, 0 \leqslant i \leqslant l-1}^{N} (-1)^i \frac{\langle (X-k)^i u_k, \varphi \rangle}{(z+k)^{i+1}}$$

$+$ holomorphic on $\operatorname{Re} z \leqslant N$.

Dynamical definition of residue

Suppose $\Gamma|_\Delta \subset N^*\Delta$. Let $\Pi_0 :=$ projection on zero resonance.

The dynamical residue of $\mathcal{K}$ (w.r.t. X) is:

$$\mathrm{res}_X \mathcal{K} = \iota_\Delta^* \big(X(\Pi_0(\mathcal{K})) \big) \in C^\infty(M).$$

Might be ill-defined, and might depend on X. But...

Theorem (Dang–Wrochna '21)

If $A \in \Psi^\alpha_{\mathrm{ph}}(M)$, $\alpha \in \mathbb{C}$, then for every radial vector field X,

$$\mathrm{res}\, A = (\mathrm{res}_X \mathcal{K}_A) d\mathrm{vol}_g.$$

techniques related to Connes–Moscovici '95, Kontsevich–Vishik '95, Joshi '97, Lesch '99, Lesch–Pflaum '00, Paycha '07

II. Lorentzian complex powers

Lorentzian spectral ζ-function density

Assume (M, g) is a globally hyperbolic perturbation of Minkowski space $(\mathbb{R}^n, g^0 = dx_0^2 - dx_1^2 - \cdots - dx_{n-1}^2)$ (or more general non-trapping Lorentzian scattering space), and its dimension n is even.

Theorem (Dang–Wrochna '20)

For $\varepsilon > 0$, the Schwartz kernel of $(\Box_g - i\varepsilon)^{-\alpha}$ has for Re $\alpha > \frac{n}{2}$ *an on-diagonal restriction $(\Box_g - i\varepsilon)^{-\alpha}(x,x)$, which extends as a meromorphic function of $\alpha \in \mathbb{C}$ with poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, 1\}$. Furthermore,*

$$\lim_{\varepsilon \to 0^+} \operatorname{res}_{\alpha = \frac{n}{2} - 1} \underbrace{(\Box_g - i\varepsilon)^{-\alpha}(x,x)}_{=:\zeta_{g,\varepsilon}(\alpha)} = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2} - 1\right)},$$

where $R_g(x)$ is the scalar curvature at $x \in M$.

— Essential self-adjointness of $\Box_g$ due to Vasy '19 and generalized by Nakamura–Taira '19–'22 , cf. Gérard–Wrochna '19 + Taira '22 for interpretation via asymptotics at $t \to \pm\infty$.

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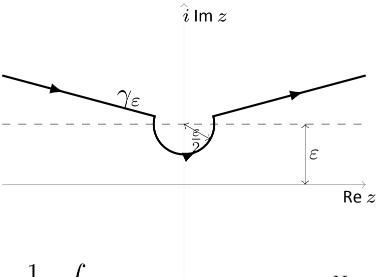
— **Spectral action for gravity!** Proof directly in Lorentzian signature. Perturbations of Minkowski included (no symmetries assumed).

— The $\varepsilon \to 0^+$ avoids low-frequency problems and responsible for relationship with Feynman propagator.

What is the local structure of $\zeta_{g,\varepsilon}(\alpha)$?

1) Let $P = \Box_g$. Express $(P - i\varepsilon)^{-\alpha}$ as contour integral of $(P - z)^{-1}$.
   For $\alpha = N + \mu > 0$:



$$(P - i\varepsilon)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\mu} (P - i\varepsilon)^{-N} (P - z)^{-1} dz, \quad .$$

2) Construct a (Feynman type) Hadamard parametrix $H_N(z)$ and show it approximates the resolvent uniformly in z.

3) Study scaling properties, compute poles and get curvature R_g from contour integrals of $H_N(z)$.

Construction of Hadamard parametrix $H_N(z)$

— Let $\mathbf{F}_\alpha(z,.)$ be locally given by

$$F_\alpha(z,x)=\frac{1}{\Gamma(\alpha+1)(2\pi)^n}\int e^{i\langle x,\xi\rangle}\left(|\xi|^2_{g^0}-i0-z\right)^{-\alpha-1}d^n\xi$$

(in normal coordinates) then ansatz of order N:

$$H_N(z,.)=\sum_{k=0}^{N}u_k\mathbf{F}_k(z,.)\in\mathcal{D}'(\mathcal{U}).$$

solved modulo errors by transport equations thanks to

$$(P-z)\,(u\mathbf{F}_\alpha)=\alpha u\mathbf{F}_{\alpha-1}+(Pu)\mathbf{F}_\alpha+(hu+2\rho u)\frac{\mathbf{F}_{\alpha-1}}{2}$$

for all $u\in C^\infty(M)$, where $h(x)=b^j(x)g^0_{jk}x^k$ and $\rho=x^k\partial_{x^k}$.

— Hölder–Zygmund and microlocal estimates for $F_\alpha(z,.)$

competition between regularity in x and decay in z

- Hadamard parametrix variants used in QFT; analytic continuation of eigenfunctions Zelditch ’18; Lorentzian local index theory Bär–Strohmaier ’20. We followed mostly Hörmander vol. 3, Sogge ’14 and Zelditch.

- z-dependent Hadamard in compact Riemannian setting used by Sogge ’88, Dos Santos Ferreira–Kenig–Salo ’14, Bourgain–Shao–Sogge–Yao ’15

Hadamard parametrix H_N approximates $(P-z)^{-1}$?

$$(P-z)\left(\sum_{k=0}^{N} u_k \mathbf{F}_k(z,.)\chi\right) = |g|^{-\frac{1}{2}}\delta_\Delta + (Pu_N)\mathbf{F}_N(z,.)\chi + r_N(z),$$

where Pu_N highly regular, and r_N singular (but 0 near diagonal). Applying $(P-z)^{-1}$ well-defined and yields good errors if $(P-z)^{-1}$ is shown to have special structure of *singularities* and mapping properties uniformly in z.

More precisely, we need $(P-z)^{-1}$ to have Feynman wavefront set, uniformly in Im $z > 0$.

> This means microlocally same singularities as $F_0(z,x)$. Proofs use radial estimates in sc-calculus (Melrose ’94, Vasy et al. ’13–’19).

Main result

Theorem (Dang–Wrochna ’20)

Let (M,g) be Lorentzian of even dimension n, and suppose $\Box_g$ has Feynman resolvent uniformly in Im $z>0$. *For all radial X and all $k=1,\dots,\frac{n}{2}$ and $\varepsilon>0$,*

$$\operatorname*{res}_{\alpha=k}\zeta_{g,\varepsilon}(\alpha)=\frac{1}{2}\mathrm{res}_X\big((\Box_g-i\varepsilon)^{-k}\big),$$

where $\zeta_{g,\varepsilon}(\alpha)$ is the spectral zeta function density of $\Box_g-i\varepsilon$.

“Analytic residues of $\zeta_{g,\varepsilon}$ are dynamical residues (scaling anomalies).”

Note: $(P-i\varepsilon)^{-\alpha}(x,x)$ expressed by contour integrals of $\mathbf{F}_\beta(z,.)$.

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z-i\varepsilon)^{-\alpha}\mathbf{F}_k(z,.)dz=\frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)}\mathbf{F}_{k+\alpha-1}(i\varepsilon,.)$$

How about Dirac operators?

The Lorentzian Dirac operator D_g satisfies $D_g^2 = \Box_g +$ l.o.t. in vector bundle sense. But it is in general **not** formally self-adjoint in any reasonable sense.

- ✓ Local analysis of Hadamard parametrix generalizes to $P = D_g^2$.
- ✓ Dynamical residue also understood for $Q(P - z)^{-1}$.
- *work in progress* with Nguyen Viet Dang (Jussieu) & András Vasy (Stanford): does D_g on Lorentzian scattering space (M, g) have Feynman resolvent for $z \in \mathbb{C}$ in reasonable sector ?

III. Summary and outlook

To sum up...

- Dynamical definition of residue coincides with Guillemin–Wodzicki residue for ΨDOs.
- Spectral zeta function density $\zeta_{\varepsilon,g}$ makes sense in Lorentzian setting, and its poles contain local geometric information!
  - $\Rightarrow$ **Gravity can be derived from a spectral action**. The formulation uses notions natural in Quantum Field Theory.
- Analytic residues of $\zeta_{g,\varepsilon}$ are dynamical residues.

Lorentzian setting is more singular. But singularities are *good* (cf. inverse problems), and special role of null geodesic flow (and its asymptotic structure, cf. horizons) makes it a rich source of problems!

Thank you for your attention!

IV. Appendix

Suppose $P = \partial_t^2 - \Delta$, Im $z > 0$. Retarded propagator of $P - z$:

$$\theta(t-s)\frac{e^{i(t-s)\sqrt{-\Delta-z}} - e^{-i(t-s)\sqrt{-\Delta-z}}}{2i\sqrt{-\Delta - z}}$$

Looks like no chance of $\|(P-z)^{-1}\| \leq |\text{Im}\, z|^{-1}$. But:

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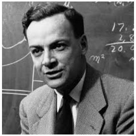

"Every particle in Nature has an amplitude to move backwards in time, and therefore has an anti-particle."

– Richard Feynman

$$\big((P-z)^{-1}u\big)(t,.) = -\frac{1}{2}\int \frac{e^{-i|t-s|\sqrt{-\Delta-z}}}{\sqrt{-\Delta-z}}u(s,.)ds. \qquad (1)$$

The boundary value $(P - i0)^{-1}$ is the Feynman propagator.

For general $P = \Box_g = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu)$, however, we can only hope:

$$(P-z)^{-1} = U_z^+ + U_z^- + z\text{-dependent error}$$

Start with (1) at infinity, then propagate!

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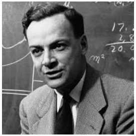

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- Use propagation estimates (introduced in GR by Vasy '13, also used in non-linear Kerr–de Sitter stability Hintz–Vasy '18 and linear Kerr stability Häfner–Hintz–Vasy '21)

Lorentzian scattering spaces

Example: Minkowski metric $g^0 = dx_0^2 - (dx_1^2 + \cdots + dx_{n-1}^2)$ on $\mathbb{R}^n$ extends to radial compactification $\overline{\mathbb{R}}^n$ defined using boundary-defining function $\rho = (x_0^2 + x_1^2 + \cdots + x_n^2)^{-\frac{1}{2}}$. Regularity w.r.t. $\rho^2 \partial_\rho = -\partial_r$

Definition: Lorentzian sc-metrics are C^∞ sections of ${}^{\mathsf{sc}}T^*M \otimes_{\mathsf{s}} {}^{\mathsf{sc}}T^*M$, where ${}^{\mathsf{sc}}T^*M$ generated by $\rho^{-2}d\rho$, $\rho^{-1}dy_1, \ldots \rho^{-1}dy_{n-1}$.

Null geodesics lift to null bicharacteristics on ${}^{\mathsf{sc}}T^*M$ (rescaled and extended at ∂M appropriately)

Definition: (M, g) non-trapping Lorentzian sc-space if there are sinks/sources $L_\pm$ above ∂M, and null bicharacteristics flow from and to L_- and L_+.

Includes small **perturbations of Minkowski space** and **asymptotically Minkowski** spaces.

Theorem (Vasy '20)

$\Box_g$ *is essentially self-adjoint on* $C_{\mathsf{c}}^\infty(M)$ *in* $L^2(M)$.

(In particular $L^2(M)$ *solvability of* $(\Box_g - i\varepsilon)u = f$ *for* $f \in C_{\mathsf{c}}^\infty(M)$*.)*

Assume (M, g) is a globally hyperbolic, non-trapping Lorentzian scattering space, and its dimension n is even.

Theorem

For $\varepsilon > 0$, the Schwartz kernel of $(\Box_g - i\varepsilon)^{-\alpha}$ has for Re $\alpha > \frac{n}{2}$ *a well-defined on-diagonal restriction $(\Box_g - i\varepsilon)^{-\alpha}(x, x)$, which extends as a meromorphic function of $\alpha \in \mathbb{C}$ with poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, 1\}$. Furthermore,*

$$\lim_{\varepsilon \to 0^+} \operatorname*{res}_{\alpha = \frac{n}{2} - 1} (\Box_g - i\varepsilon)^{-\alpha} (x, x) = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} - 1\right)}.$$

Theorem

For any Schwartz f with Fourier transform in $]0, +\infty[$,

$$f\left((\Box_g + i\varepsilon)/\lambda^2\right)(x, x) = \sum_{j=0}^{N} \lambda^{n-2j} C_j(f)\, a_j(x) + \mathcal{O}(\varepsilon, \lambda^{n-2N-1}),$$

where $a_j(x)$ are related to Hadamard coefficients.

In particular $a_0(x) = (4\pi)^{-\frac{n}{2}}$, $C_0(f) = i^{-1} e^{\frac{in\pi}{4}} \int_0^\infty \widehat{f}(t) t^{\frac{n}{2}-1} dt$
$a_1(x) = -(4\pi)^{-\frac{n}{2}} \frac{1}{6} R_g(x)$, $C_1(f) = i^{-1} e^{\frac{i(n-2)\pi}{4}} \int_0^\infty \widehat{f}(t) t^{\frac{n}{2}-2} dt$.

Proof of remaining crucial part

1. Estimates for $(P - i\varepsilon)^{-\alpha}$ in weighted Sobolev spaces

— Radial estimates for $(P-z)^{-1}$ following Vasy '20. For weight orders s.t. $\ell > -\frac{1}{2}$ at L_- and $\ell < -\frac{1}{2}$ at L_+:

$$\|u\|_{s,\ell} + (\operatorname{Im} z)\|u\|_{s-\frac{1}{2},\ell+\frac{1}{2}} \leqslant C(\|(P-z)u\|_{s-1,\ell+1} + \|u\|_{S,L}),$$

Melrose '93, Vasy '13, Gell-Redman–Haber–Vasy '16, Vasy '20, ...

— We get $(P-z)^{-N} : L^2_{\mathrm{c}}(M) \to H^N_{\mathrm{loc}}(M)$. Wavefront set uniform in z?

2. Evolutionary parametrix

— Feynman parametrix for $P - z$ of the form

$$U^+(z) + U^-(z), \quad U^\pm(z) = O(\langle z\rangle^{-1}) \text{ along } \gamma_\varepsilon,$$

where $U^\pm(z)$ solves (1^{st} order in time) forward/backward problem generated by ΨDO with positive/negative principal symbol. Uniform wavefront set by argument like Egorov theorem.

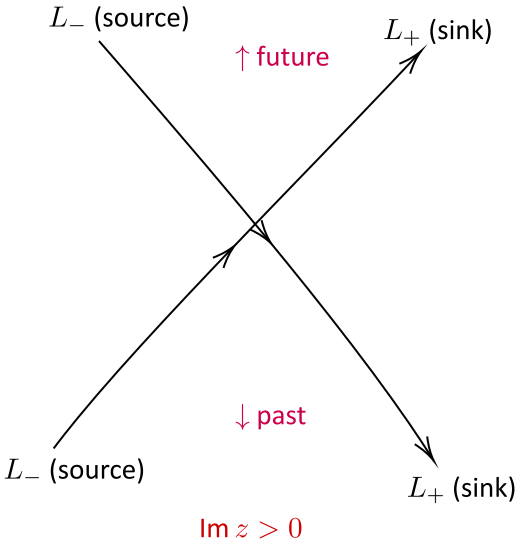
- z-dependent calculus of Shubin '01, parametrix similar to Gérard–Wrochna '19

3. Identity $U^+(z) + U^-(z) = (P-z)^{-1}$ modulo C^∞

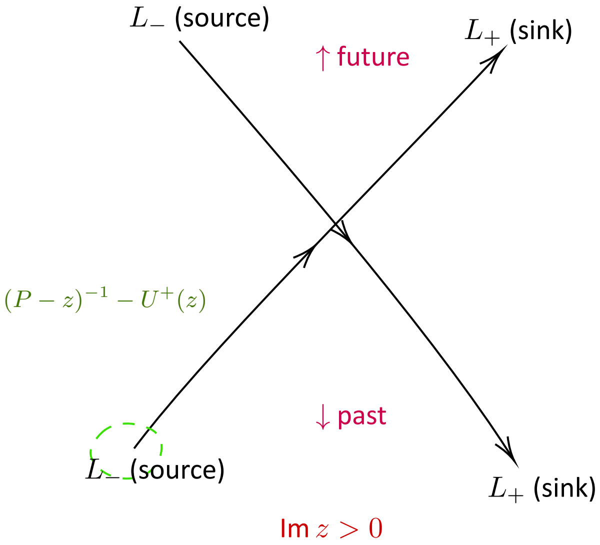
— We use radial estimates.

— *Conclusion:* WF$\big((P-z)^{-1}\big)$ estimated uniformly and involutive

- problem of computing WF$\big((P-z)^{-1}\big)$ analogous to Dyatlov–Zworski '16, but our strategy closer to Vasy–Wrochna '18



L_- (source)
L_+ (sink)
↑ future
↓ past
L_- (source)
L_+ (sink)
Im $z > 0$


L_- (source)
$\uparrow$ future
L_+ (sink)
$(P-z)^{-1} - U^+(z)$
$\downarrow$ past
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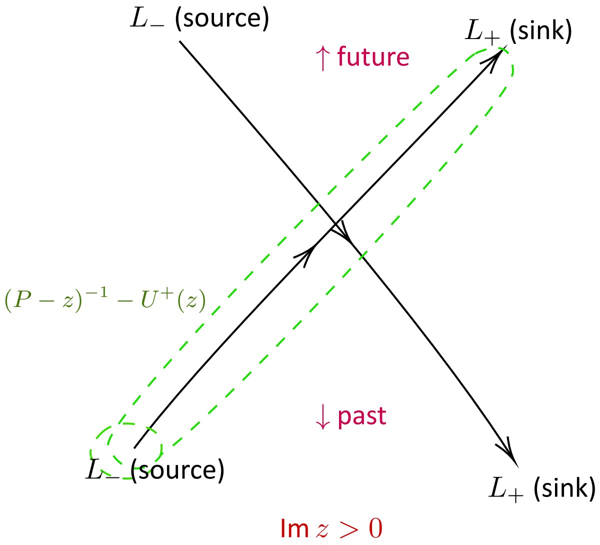
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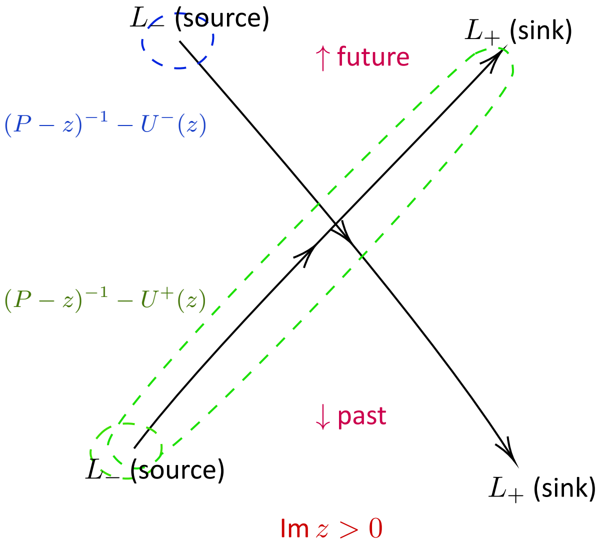

L_- (source)
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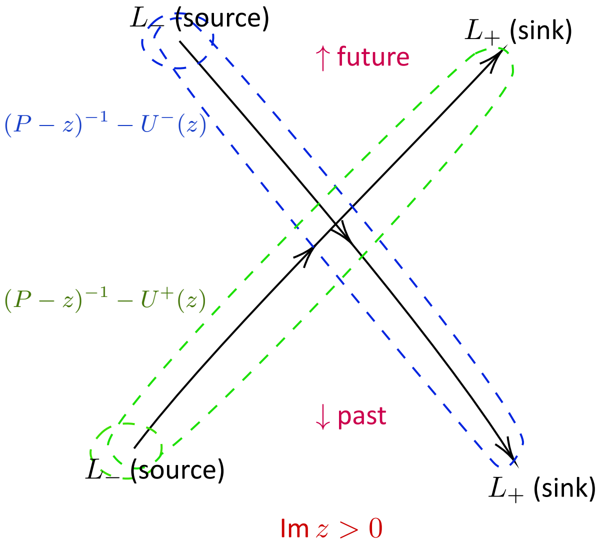

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