On the L^p Spectrum of the Dirac operator

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Introduction

We consider a *Clifford bundle* $S \to M$ over a smooth Riemannian noncompact manifold M^n.

We refer to the sections of S as *spinors*.

∇ is a metric connection on S, compatible with the Levi-Civita connection on M, s.t. the Clifford action is skew-adjoint with respect to the fiberwise Hermitian metric $\langle ., . \rangle$.

D_c is the associated Dirac operator on the set of $C_c^\infty(S)$.

$\not{D}$ is the *classical Dirac operator* when M is spin and S is the associated spinor bundle.

D_c extends to a self-adjoint operator on L^2, but also to a closed operator on L^p.

Define D_p as the completion of $D_c \colon C_c^\infty(S) \subset L^p(S) \to C_c^\infty(S) \subset L^p(S)$ with respect to the graph norm as an operator from L^p to L^p, for $p \in [1, \infty)$.

Introduction

We are interested in the L^p spectrum of a general Dirac operator and its square.

The *spectrum* of an operator $\mathcal{L}$, consists of all points $\lambda \in \mathbb{C}$ for which

$$\mathcal{L} - \lambda I$$

fails to be invertible on its domain.

It is a closed set.

The definition for the spectrum is the same for any self-adjoint operator H on a Hilbert space $\mathcal{H}$, and also for a closed operator on a Banach space.

• We denote the spectrum of a geometric operator $\mathcal{L}_p$ (i.e. Dirac, Laplacian), with domain in L^p by $\sigma(\mathcal{L}_p)$.

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Observations:

• For any self-adjoint nonnegative operator $\sigma(\mathcal{L}) \subset [0, \infty)$.

• For an operator on a Banach space the spectrum could contain points in the complex plane that are outside the real axis.

• In general, $\sigma(\mathcal{L}_p) = \sigma(\mathcal{L}_{p^*})$ whenever $1/p + 1/p^* = 1$ by duality, with $\mathcal{L}_\infty = (\mathcal{L}_1)^*$.

• The spectrum of geometric operators like the Dirac operator or the Laplacian reflects the geometric structure and analytic properties of the space and the bundle.
Computing the spectrum is in general a difficult task which requires various assumptions on the symmetries and structure of the manifold.

• In the case that the manifold is compact, the L^p spectrum of the Dirac operator (and the Laplacian) is always a discrete set and is the same for all p, hence always contained in the real line.
Therefore, studying the L^p spectrum for $p \neq 2$ is only interesting in the noncompact case.

• For many geometric operators, $\sigma(\mathcal{L}_p)$ will contain a region in the complex plane when curvature is negative at an end of the manifold.

• Ammann and Große, 2016 have shown that on manifolds that are products of hyperbolic space and compact manifolds the L^p spectrum of the classical Dirac operator depends on p, and contains a region in the complex plane.

The region is a set of points z such that z^2 lies inside a parabola.

For $p = 2$ it reduces to an interval in the real line.

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- A similar picture is true for the Laplacian on L^p integrable k-forms, $\Delta_{k,p}$.

- In the function case $\sigma(\Delta_{0,p})$ contains a parabolic region for $p \neq 2$ over

- the hyperbolic space $\mathbb{H}^n$ and Kleinian groups, $M = \mathbb{H}^n/\Gamma$, with Γ a geometrically finite group of isometries, Davies, Simon and Taylor 1988.

- symmetric spaces of non-compact type, Lahoué and Rychener, and Taylor in the 1980's.

- locally symmetric spaces, Ji and Weber 2007-10.

- In the k-form case $\sigma(\Delta_{k,p})$ is a parabolic region over $\mathbb{H}^n$ depending on p (C.- Z. Lu).

Introduction

• On the other hand, it is possible to prove the L^p independence of the spectrum for the Laplacian (and Schrödinger operators) under certain conditions on the curvature and geometry of the manifold.

• The L^p independence of the spectrum for Schrödinger operators over $\mathbb{R}^n$ was proved by Hempel and Voigt 1986-87 depending on the growth of the negative part of the potential. The L^p independence result relies on the existence of Gaussian-type heat kernel estimates.

• Kordyukov 1991 generalized this result to uniformly elliptic operators with uniformly bounded smooth coefficients over a manifold of bounded geometry with subexponential volume growth.

• Sturm 1992 proved that if the manifold has Ricci curvature bounded below and uniformly subexponential volume growth, then the L^p spectrum is independent of p for a class of uniformly elliptic operators in divergence form.

-Both results apply to the Laplacian on functions.

• For the Laplacian on k-forms it was proved that $\sigma(\Delta_{k,p})$ is independent of p if, in addition to Sturm's geometric conditions, the Weitzenböck tensor $\mathcal{W}_k$ is bounded below (C.).

• One of the important aspects of L^p independence results is that they allow to find a large class of manifolds over which the L^p spectrum is contained in the real line, and the L^2 spectrum is maximal.

It is easier to compute the L^1-spectrum of an operator because L^1 estimates of test functions are easier to obtain in comparison to L^2 estimates.

• For example Sturm's L^p independence result was used by J. Wang 1997, and Z. Lu- D. Zhou 2011 to show that $\sigma(\Delta_{0,p}) = [0, \infty)$ on manifolds with asymptotically nonnegative Ricci curvature.
For $p = 2$ this gives us that the spectrum is maximal.

Introduction: Aims

• We are interested in finding sufficient conditions on the manifold such that the L^p spectrum of the Dirac operator and the square Dirac operator is p-independent. This would allow us to have a large class of manifolds where the L^p spectra are equal and $\sigma(D_p^2) = [0, \infty),\ \sigma(D_p) = \mathbb{R}$.

• L^p independence results rely on Gaussian heat kernel estimates.
There is a well defined heat semigroup for square Dirac operators, but not for the Dirac operator itself.

We prove a L^p independence result for the square Dirac operator, which extends to the Dirac operator on L^p whenever the manifold has the right dimension or certain symmetric structures.

Introduction: Aims

• We are also interested in finding large classes of manifolds when the L^2 spectrum of the Dirac and square Dirac operator is maximal.

• Previous examples only considered the classical Dirac operator, and in very particular (albeit interesting) settings.

• Kawai in 2008,16 proved that $\sigma(\not{D}_2) = \mathbb{R}$ on spin manifolds with an end where the metric can be written in radial polar coordinates outside a compact set and sectional curvature decaying in absolute value at a rate $c\, r^{-2}$.

• Bär in 2000 proved that over hyperbolic manifolds with finite volume $\sigma(\not{D}_2) = \mathbb{R}$ if the spin structure is trivial along at least one cusp.
The spin structure affects the spectrum: If the spin structure is nontrivial along all cusps, then $\sigma(\not{D}_2)$ is discrete.

• We use both the L^p independence result, and the generalized Weyl criterion (C.-Lu 2014) to find large classes of manifolds where $\sigma(D_2) = \mathbb{R}$ and $\sigma(D_2^2) = [0, \infty)$.

L^p Independence: Heat kernel estimates

• The square Dirac operator, D^2, satisfies the Weitzenböck formula

$$D^2 = \nabla^* \nabla + \mathcal{R}.$$

where $\mathcal{R} \in \mathrm{End}(S)$ is the *Clifford contraction* which acts as a tensor on spinors.

• When M is spin, and $\not{D}$ is the classical Dirac operator, $\mathcal{R} = \mathrm{scal}/4$ where scal is the scalar curvature of M^n.

• For an oriented manifold the complexified exterior bundle with its natural metric and connection is a Clifford bundle, and $D = d^* + d$.
$\mathcal{R}$ is determined by the curvature tensor of M (for 1-forms $\mathcal{R} = \mathrm{Ric}$).

L^p Independence: Heat kernel estimates

• To obtain heat kernel estimates for D_2^2, we use a domination technique:

• Use the Weitzenböck formula and Kato's inequality, $|d|\varphi|| \leq |\nabla\varphi|$, to show that for any $\varphi \in C_c^\infty(S)$

$$\mathrm{Re}\langle D^2\varphi, \varphi\rangle = |\varphi|\Delta|\varphi| - |d|\varphi||^2 + |\nabla\varphi|^2 + \langle \mathcal{R}(\varphi), \varphi\rangle \geq |\varphi|(\Delta - K_1)|\varphi|$$

whenever the Clifford contraction satisfies $\mathcal{R} \geq -K_1$, where Δ is the Laplacian on functions.

• By Hess, Schrader and Uhlenbrock 1977, we can dominate the semigroup corresponding to D^2 on L^2, by the heat semigroup of the Laplacian on L^2 integrable functions: For any for any $\varphi \in \mathrm{dom}(D_2^2) \subset L^2(S)$

$$\left| e^{-tD_2^2}\varphi \right| \leq e^{-t(\Delta_2 - K_1)} |\varphi| .$$

L^p Independence: Heat kernel estimates

Lemma 1

Whenever M is a manifold with Ricci curvature bounded below $\mathrm{Ric} \geq -K_0$, $\mathcal{R} \geq -K_1$, the heat operator e^{-tD^2} on $L^2(S) \cap L^p(S)$ can be extended to a bounded operator on $L^p \to L^p$ for all $p \in [1, \infty)$.

For $p = \infty$, e^{-tD^2} is defined as the dual of the operator on L^1.

Even though it is not necessarily a contraction on L^p, for all $t \geq 0$ and $p \in [1, \infty]$ we have the upper bound

$$\|e^{-tD^2}\|_{p \to p} \leq e^{K_1 t}.$$

- This allows us to define the infinitesimal generator, H_p, of the semigroup e^{-tD^2} on L^p for all $1 \leq p \leq \infty$.

L^p Independence: Heat kernel estimates

• Technical results: We are able prove that for $1 \leq p < \infty$ the infinitesimal generator H_p coincides—as one expects—with the square of the Dirac operator $(D_p)^2$ on L^p.
However, it is still unclear whether H_∞ coincides with $(D_1^*)^2$ in general.

• Denote H_p by D_p^2 for all $1 \leq p \leq \infty$ from now on.

• The domination property also implies the existence of a heat kernel for D_2^2, $\vec{H}(x, y, t)$, and turns into a pointwise domination:

$$\left|\vec{H}(x, y, t)\right| \leq h(x, y, t)\, e^{K_1 t}$$

where $h(x, y, t)$ is the heat kernel of the Laplacian on functions.

L^p Independence: Heat kernel estimates

Proposition 1

Under the assumptions of Lemma 1 we get:

$$\left|\vec{H}(x,y,t)\right| \leq C_1 \, V^{-1/2}(x,\sqrt{t})\, V^{-1/2}(y,\sqrt{t}) \cdot \exp\left[-\frac{d^2(x,y)}{C_2\, t} + C_3\sqrt{K_0\, t} + K_1 t\right] \qquad (1)$$

for some positive constants $C_i(n)$, where $V(x,r)$ is the volume of the ball of radius r centered at x, and $d(x,y)$ is the distance between x, y.

• For the classical square Dirac operator, the heat kernel estimate holds whenever $\mathrm{Ric} \geq -K_0$, with $K_1 = n\, K_0/4$.

L^p Independence of the spectrum

Theorem 2

Let M^n be a complete Riemannian manifold with Clifford bundle S and associated Dirac operator D. Suppose that $Ric \geq -K_0$, $\mathcal{R} \geq -K_1$, and the volume of M grows uniformly subexponentially.

Then the L^p-spectrum of the square Dirac operator acting on L^p, D^2_p, is independent of p for all $p \in [1, \infty]$. Moreover, the set of isolated eigenvalues of finite multiplicity and their algebraic multiplicity is independent of p for all $p \in [1, \infty]$.

If additionally $\sigma(D_2)$ is symmetric and $\sigma(D_1) \neq \mathbb{C}$, then the L^p-spectrum of D_p is independent of p for all $p \in [1, \infty)$.

Uniformly subexponential volume growth: For any $\varepsilon > 0$ and $x \in M$ there exists a constant $C(\varepsilon)$, such that for any $r > 0$

$$V(x, r) \leq C(\varepsilon)\, V(x, 1)\, e^{\varepsilon r}.$$

It implies nonexponential growth and nonexponential decay.

L^p Independence of the spectrum

Sketch of Proof:

• The kernel of the resolvent operator at $\alpha < 0$ can be controlled by the heat kernel via

$$(D_2^2 - \alpha)^{-m/2} = C_m \int_0^\infty e^{-tD_2^2}\, t^{\frac{m}{2}-1}\, e^{\alpha t}\, dt.$$

• Let ξ is in the resolvent set of D_2^2.
Use the resolvent equation (to recenter the resolvent operator around ξ).

The Gaussian heat kernel estimate, together with the uniformly subexponential volume growth allow us show that $(D_2^2 - \xi)^{-m}$ has a smooth L^1 integrable kernel function $G_\xi(x, y)$ for any integer m large enough.

• This implies that $(D_2^2 - \xi)^{-m}$ is bounded from L^p to L^p for all $1 \leq p \leq \infty$, and it must therefore be equal to $(D_p^2 - \xi)^{-m}$, whenever ξ is in the resolvent set of D_2^2.

• Therefore ξ is also in the resolvent set of D_p^2.
i.e. the L^p spectrum is contained in the L^2 spectrum.

L^p Independence of the spectrum

• The other containment is an analytic result that follows from interpolation theory.

• For the L^p-independence of the multiplicity of isolated eigenvalues: Consider the Laurent series expansion for the resolvent operator $(D_p^2 - \xi)^{-1}$ on a small disc around the isolated eigenvalue λ. They are the same for all p, by L^p-independence of the spectrum.

The order of the pole at λ is the multiplicity of the eigenvalue, hence the multiplicities are the equal.

L^p Independence of the spectrum for D_p

From the L^p independence for the spectrum of D_p^2, to that of D_p:

- This is the only statement where $\sigma(D_1) \neq \mathbb{C}$ is used.

- By the Riesz-Thorin interpolation theorem $\sigma(D_2) \subset \sigma(D_p) \subset \sigma(D_1)$ for all $1 \leq p \leq 2$. As a result, $\sigma(D_p) \neq \mathbb{C}$ for all $p < \infty$ (by duality).

- Key result: If $\sigma(D_p)$ is not the entire complex plane, then $\lambda^2 \in \sigma(D_p^2)$ if and only if λ or $-\lambda$ belongs to $\sigma(D_p)$ (generalization of Ammann and Große, 2016 for $\not{D}$).

- Consider a point $\lambda \in \sigma(D_p)$. Then, $\lambda^2 \in \sigma(D_p^2) = \sigma(D_2^2)$, by L^p-indep.
Since $\sigma(D_2)$ is assumed to be symmetric, $\lambda \in \sigma(D_2)$.
Hence, $\sigma(D_p) \subset \sigma(D_2)$.

- The Riesz-Thorin interpolation also implies the converse inclusion $\sigma(D_2) \subset \sigma(D_p)$ for all $p < \infty$, and we get, $\sigma(D_p)$ is independent of p.

- If M and the Clifford bundle are of bounded geometry, the assumption $\sigma(D_1) \neq \mathbb{C}$ is automatically fulfilled (Ammann and Große). For the manifolds considered in Theorem 2 this *should* also be the case.

L^p Independence of the spectrum for $\not{D}_p^2$, $\not{D}_p$

- For the classical Dirac operator:

Corollary 3

Let M^n be a spin Riemannian manifold with Ricci curvature bounded below, and uniformly subexponential volume growth.
Then the L^p-spectrum of $\not{D}_p^2$ is independent of p for all $p \in [1, \infty]$.

If additionally $\sigma(\not{D}_1) \neq \mathbb{C}$ and if $n \neq 3$ modulo 4 or M has an orientation-reversing isometry that lifts to spin structure, then the L^p-spectrum of $\not{D}_p$ is independent of p for all $p \in [1, \infty)$.

For the classical Dirac operator: Whenever dim $M \neq 3$ modulo 4, or if M admits an orientation reversing isometry that lifts to the spin structure, $\sigma(\not{D}_p)$ is always symmetric (Ammann Große 2016).

Computation of the L^p spectrum of D_p^2, D_p

- The computation of the spectrum always requires the construction of a large class of approximate eigenspinors, or test spinors.

- First goal: Find manifolds such that $\sigma(D_p^2) = [0, \infty)$ and $\sigma(D_p) = \mathbb{R}$ for all p.

- Idea: Show the result for $p = 1$ and use the L^p-independence result.

Theorem 4

Let $(\tilde{M}^n, \tilde{g})$ be a complete spin Riemannian manifold with uniformly subexponential volume growth with one end isometric to
$M = N \times [0, \infty)$, $g = f(t)^2 g_N + dt^2$, *and s.t.* $\ker \not{D}^N \neq \varnothing$.
Assume f is positive, monotonically increasing with $\lim_{t\to\infty} \frac{f^{n-1}(2t)}{t f^{n-1}(t)} \to 0$.

Then, $\sigma(\not{D}_p^2) = [0, \infty)$ *for all* $p \in [0, \infty]$.

If additionally $\dim \tilde{M} \neq 3$ *mod 4 or M admits an orientation reversing isometry that lifts to the spin structure, then, the* $\sigma(\not{D}_2) = \mathbb{R}$.
If we also know that $\sigma(\not{D}_1) \neq \mathbb{C}$, *then* $\sigma(\not{D}_p) = \mathbb{R}$ *for all* $p \in [1, \infty)$.

Computation of the L^p spectrum of D_p^2, D_p

• Admissible functions f include some functions with quasi-polynomial growth such as $f^{n-1}(t) = e^{\frac{1}{4}(\ln t)^2}$, for which it is not possible to directly compute the L^2-spectrum.

• We use the $\varphi \in \ker \not{D}^N$ to construct approximate eigenspinors $\psi_T = \eta_T \exp(-\lambda \mathrm{i} t)\, \varphi$ for $\lambda \in \mathbb{R}$ where η_T is an adequate cut-off function.

• The Dirac operator on the end M has a nice decomposition which together with the volume decay properties of M allows to show $\frac{\|(\not{D}^M - \lambda)\psi_T\|_{L^1}}{\|\psi_T\|_{L^1}} \to 0$ for a sequence of $T \to \infty$.

• Then $\mathbb{R} \subset \sigma(\not{D}_1^M)$ and hence, $[0, \infty) \subset \sigma((\not{D}_1^M)^2)$.
$\sigma(\not{D}_p^2)$ is p-independent, therefore $\sigma(\not{D}_p^2) = [0, \infty)$ for $1 \leq p \leq \infty$.

-The result for $\sigma(\not{D}_p)$ is a result the symmetry assumptions.

Computation of the spectrum of $D_p^2,\ D_p$

- Manifolds with asymptotically D^2-harmonic spinors.

Theorem 5

Let (M, g) be a complete Riemannian manifold with Clifford bundle S, such that the Clifford contraction $\mathcal{R}$ is bounded below and $Ric(x) \geq -\delta(n) r^{-2}(x)$ ($r(x)$ is the distance to a fixed point x_o and $\delta(n)$ is small).

Moreover, assume that for all $R > 0$ large enough there is a smooth section φ_R of S with support $M \setminus B_{x_o}(R)$ such that $0 < c_1 < |\varphi_R| < c_2$, $|\nabla \varphi_R| \leq C$, and $\|D^2 \varphi_R\|_{L^\infty} \to 0$ as $R \to \infty$.

Then, $\sigma(D_p^2) = [0, \infty)$ for all $p \in [1, \infty]$.
If additionally, $\sigma(D_1) \neq \mathbb{C}$ and $\sigma(D_2)$ is symmetric, then $\sigma(D_p) = \mathbb{R}$ for all $p \in [1, \infty)$.

- Under these curvature assumptions we have smooth test functions of disjoint compact support such that for any R large enough
$\|\Delta \eta_R - \lambda \eta_R\|_{L^1} \leq (c/R)\, \|\eta_R\|_{L^1}$ and $\|\nabla \eta_R\|_{L^1} \leq (c/R)\, \|\eta_R\|_{L^1}$.

Computation of the spectrum of D_p^2, D_p

• Using them, we can prove that $\dfrac{\|(D^2-\lambda)(\eta_R\varphi_R)\|_{L^1}}{\|\eta_R\varphi_R\|_{L^1}} \to 0$ as $R \to \infty$.

• We get the spectrum of D_p^2 and D_p as in the previous theorem.

• For the classical Dirac operator over a spin manifold: Under the same assumptions on the curvature of M and the existence of an asymptotically harmonic spinor, we get $\sigma(\not{D}_p^2) = [0,\infty)$ for all $p \in [1,\infty]$.

For $\not{D}_p$: If in addition $n \neq 3$ modulo 4 or M has an orientation-reversing isometry that lifts to spin structure, then $\sigma(\not{D}_2) = \mathbb{R}$.
If we also know that $\sigma(\not{D}_1) \neq \mathbb{C}$, then $\sigma(\not{D}_p) = \mathbb{R}$ for all $p \in [1,\infty)$.

Maximal L^2 spectrum

Theorem 6 (Generalized Weyl criterion, C. -Lu, 2014)

Let (M, g) be a complete Riemannian manifold with Clifford bundle S and associated Dirac operator D. Assume that for $\lambda > 0$ there exists a sequence of sections of the Clifford bundle $\psi_i \in L^1 \cap L^\infty$ in the domain of D_1^2 such that

$$\|\psi_i\|_{L^2}^2 = 1 \quad \text{and} \quad \|\psi_i\|_{L^\infty} \|(D^2 - \lambda)\psi_i\|_{L^1} \to 0 \;\; \text{as} \;\; i \to \infty.$$

Then $\lambda \in \sigma(D_2^2)$.

• In other words, if we have a sequence of *bounded* test spinors such that $\|\psi_i\|_{L^2} = 1$ and $\|(D^2 - \lambda)\psi_i\|_{L^1} \to 0$, then λ is in the spectrum.

• The criterion only works for self-adjoint nonnegative operators, hence it cannot by applied to the Dirac operator.

Maximal L^2 spectrum

Theorem 7

Suppose that $\mathrm{Ric}(\partial_r, \partial_r) \geq -(n-1)\delta(r)$, *where* $r(x)$ *is the distance to a fixed point* x_o, ∂_r *is the radial direction, and* $\delta(r) \to 0$ *as* $r \to \infty$.
If the volume of M *is finite, we also assume that its volume does not decay exponentially at* x_o.

In addition, suppose that there exists a family of sections of the Clifford bundle φ_i *such that (except possibly on a compact set* K*) for all* i *large enough*
$0 < c_1 < |\varphi_i| < c_2$, $|\nabla \varphi_i| \leq C$, *and* $\|D^2\varphi_i\|_{L^\infty} \to 0$ *as* $i \to \infty$.

Then $\sigma(D_2^2) = [0, \infty)$.

• The volume of such manifolds grows subexponentially at x_o, but it need not grow uniformly subexponentially. We can prove that the L^2 spectrum is maximal, but we cannot control the L^p spectrum to show that it is only a subset of the real line.

• Our curvature assumptions allow us to use 'nice' test function
$\eta_i(x) = \chi_i(\tilde{r}/R_i)\, e^{\sqrt{-1}\sqrt{\lambda}\,\tilde{r}}$ where $\tilde{r}(x)$ is a smoothing $r(x)$.

Maximal L^2 spectrum

• For the particular case of the square of the classical Dirac operator over an asymptotically Euclidean spin manifold we can use the Witten spinor constructed by Bartnik to compute the L^2-spectrum of $\not{D}^2$ and $\not{D}$.

Definition 8

A manifold (M^n, g), $n \geq 3$, is *asymptotically flat* if its scalar curvature is in $L^1(M)$ and there is a compact set $K \subset M$ and a diffeomorphism $\Phi\colon M \setminus K \to \mathbb{R}^n \setminus B_{r_o}(0)$ such that

$$(\Phi_* g)_{ij} = \delta_{ij} + O(r^{2-n}), \quad \partial_k (\Phi_* g)_{ij} = O(r^{1-n}), \quad \partial_{kl} (\Phi_* g)_{ij} = O(r^{-n}).$$

If the manifold also has positive scalar curvature and is spin, then there exists a uniquely defined *Witten spinor* i.e. a solution to

$$\not{D}\psi = 0, \quad \lim_{|x|\to\infty} \psi = \psi_0 \quad \text{where} \quad \psi_0 \quad \text{is a constant spinor.}$$

The solution, is smooth and satisfies some strong decay estimates in its first and second order derivatives, Bartnik 1986 (also Finster and Kath 2002).

Maximal L^2 spectrum

By Theorem 7 the following result is immediate.

Proposition 2

Suppose that M^n with $n \geq 3$ is an asymptotically flat spin manifold with positive scalar curvature. Then the spectrum of $\not{D}_2^2$ is $[0, \infty)$.
If additionally $n \neq 3$ modulo 4 or M has an orientation-reversing isometry that lifts to spin structure, then $\sigma(\not{D}_2) = \mathbb{R}$.

Thank you!