An introduction on maximally hypoelliptic differential operators

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Differential operators

Definition

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator. D is called hypoelliptic if for any distribution u

$$Du \text{ is smooth} \Rightarrow u \text{ is smooth}$$

Example

∂_x on $\mathbb{R}$ is hypoelliptic but ∂_x on $\mathbb{R}^2$ isn't.

Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ such that

- $\phi(k) \to +\infty$ as $k \to +\infty$.
- for any distribution u and any $k \in \mathbb{N}$

  $$Du \text{ is in } H^k(M) \Rightarrow u \text{ is in } H^{\phi(k)}(M)$$

  Then by Sobolev Lemma, D is hypoelliptic.

If D is of order l, then $\phi(k) \leq k + l$. If $\phi(k) = k + l$, we say D is elliptic.

Elliptic operators

Theorem (Kohn, Nirenberg, Hörmander, $\cdots$)

Let D be of order l. Then TFAE

1. *for any $k \in \mathbb{N}$ and any u distribution, $Du \in H^k(M)$ implies $u \in H^{k+l}(M)$.*
2. *for any $(x, \xi) \in T^*M \backslash 0$, $\sigma(x, \xi) \neq 0$.*

Moreover, if M is compact then the above is equivalent to

3. *for any $k \in \mathbb{N}$, $D : H^{k+l}(M) \to H^k(M)$ is Fredholm.*

Theorem 19.5.1 Hormander

Kolmogorov's operator

We saw

$$\text{elliptic} \Rightarrow \text{hypoelliptic}$$

The operator

$$D = \partial_x^2 + x^2 \partial_y^2$$

D isn't elliptic but it is hypoelliptic.

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in \tilde{H}^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

implies

$$\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$$

and by recurrence

$$\tilde{H}^{2k}(\mathbb{R}^2) \subseteq H^k(\mathbb{R}^2)$$

Hence $\cap_k \tilde{H}^k(\mathbb{R}^2) = \cap_k H^k(\mathbb{R}^2) = C^\infty(\mathbb{R}^2)$

Theorem (Folland and Stein)

For any u and any k, $Du \in \tilde{H}^k(\mathbb{R}^2)$ implies $u \in \tilde{H}^{k+2}(\mathbb{R}^2)$

General Sobolev spaces

Let $X_1, \cdots, X_m$ be vector fields. We define

$$\tilde{H}^0(M) := L^2_{loc}(M)$$
$$\tilde{H}^{k+1}(M) := \{f \in \tilde{H}^k(M) : X_1(f), \cdots, X_m(f) \in \tilde{H}^k(M)\}$$

Hormander's condition for any $x \in M$,

$$X_1(x), \cdots, X_m(x), [X_i, X_j](x), [[X_i, X_j], X_k](x), \cdots$$

span T_xM. Then $\cap_k \tilde{H}^k(M) = C^\infty(M)$

Main theorem

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D be of order l. Then TFAE

1. *for any $k \in \mathbb{N}$ and any u, $Du \in \tilde{H}^k(M)$ implies $u \in \tilde{H}^{k+l}(M)$.*
2. *for any $x \in M$, $\pi \in \mathcal{T}_x^* \subseteq \hat{G}_x$ (set of irreducible unitary representations), $\tilde{\sigma}(D, x, \pi)$ is injective.*

Moreover, if M is compact then the above is equivalent to

3. *for any $k \in \mathbb{N}$, $D : \tilde{H}^{k+l}(M) \to \tilde{H}^k(M)$ is left invertible modulo compact operators.*

Previously a conjecture by Helffer and Nourrigat (1979). In 1985, they prove $1 \Rightarrow 2$ (full generality) and $2 \Rightarrow 1$ if G_x are of rank 2, (some other cases by Rothschild stein (1976))
We say D is maximally hypoelliptic if it satisfies the above. Maximally hypoelliptic implies hypoelliptic.

Group G_x

Consider

$$\mathcal{F}^1 := C^\infty(M)X_1 + \cdots + C^\infty(M)X_m$$
$$\mathcal{F}^2 := \mathcal{F}^1 + C^\infty(M)[X_i, X_j]$$
$$\mathcal{F}^3 := \mathcal{F}^2 + C^\infty(M)[[X_i, X_j], X_k]$$
$$\vdots$$

For some N, $\mathcal{F}^N = \mathcal{X}(M)$. Consider the localization $\frac{\mathcal{F}^i}{I_x\mathcal{F}^i}$, where $I_x = \{f \in C^\infty(M) : f(x) = 0\}$. The Lie algebra

$$\mathfrak{g}_x := \oplus_{i=1}^N \frac{\mathcal{F}^i}{\mathcal{F}^{i-1} + I_x\mathcal{F}^i}.$$

Example

Take $M = \mathbb{R}^2$, $\mathcal{F}^1 = \langle \partial_x, x\partial_y \rangle$. Then $\mathcal{F}^2 = \langle \partial_x, \partial_y \rangle$. One has $\mathfrak{g}_{x,y} = \mathbb{R}^2$ if $x \neq 0$ and $\mathbb{R}^3$ if $x = 0$.

Group G_x

By construction

$$[\mathcal{F}^i, \mathcal{F}^j] \subseteq \mathcal{F}^{i+j}, \quad i+j \leq N.$$

It descends

$$[\cdot,\cdot] : \frac{\mathcal{F}^i}{\mathcal{F}^{i-1} + I_x \mathcal{F}^i} \times \frac{\mathcal{F}^j}{\mathcal{F}^{j-1} + I_x \mathcal{F}^j} \to \frac{\mathcal{F}^{i+j}}{\mathcal{F}^{i+j-1} + I_x \mathcal{F}^{i+j}}.$$

Hence a graded nilpotent Lie algebra structure on

$$\mathfrak{g}_x = \oplus_{i=1}^{N} \frac{\mathcal{F}^i}{\mathcal{F}^{i-1} + I_x \mathcal{F}^i}.$$

The group G_x is the simply connected nilpotent Lie group integrating $\mathfrak{g}_x$.

Order of differential operator

Any differential operator can be written $P(X_1, \cdots, X_m)$ where P is a noncommutative polynomial. Because of Hormander's condition and that a commutator

$$[X_i, X_j] = X_i X_j - X_j X_i$$

is a polynomial in X_i, X_j.

Definition

The Hormander order of a D is the minimum degree of P such that $D = P(X_1, \cdots, X_m)$.

Remark: Hormander order $\geq$ classical order

Example

Take ∂_x and $x\partial_x$. Then ∂_x and $x\partial_y$ are of Hormander order 1 but ∂_y is of Hormander order 2.

Order of differential operator

Recall that if D is of order l, then $D : H^{k+l}(M) \to H^k(M)$ is bounded.

Proposition

If D has Hormander order l, then for any k,

$$D : \tilde{H}^{k+l}(M) \to \tilde{H}^k(M)$$

is bounded.

Symbol

We defined $\mathfrak{g}_x$ and Hormander order. We still need symbol.
Let $\pi : G_x \to B(H)$ an irreducible unitary representation. The derivative

$$d\pi : \mathfrak{g}_x \to \mathcal{L}(C^\infty(\pi)),$$

where $C^\infty(\pi) \subseteq H$ is smooth vectors.

Definition

The symbol of $D = P(X_1, \cdots, X_m)$ is given by
$\tilde{\sigma}(D, x, \pi) := P_{hom}(d\pi(X_1), \cdots, d\pi(X_m))$

Remark: if $G_x = T_x M$, $\pi = e^{i\langle \cdot, \xi \rangle}$ and so $d\pi(X) = i\xi(X)$.

Is this well defined?

No ! But

Theorem

If $\pi \in \mathcal{T}_x$. Then for any D, $\tilde{\sigma}(D, x, \pi)$ is well defined.

We call $\mathcal{T}_x$ the characteristic set.

Characteristic set

Orbit method says

Irreducible unitary representations of G_x bijectively correspond to $Ad^*(G_x)$ orbits in $\mathfrak{g}_x^*$.

An element $l \in \mathcal{T}_x^*$ if there exists $t_n \in \mathbb{R}_+^*$ and $x_n \in M$ and $\xi_n \in T_{x_n}^* M$ such that

- $x_n \to x$ and $t_n \to 0$
- Recall

$$\mathfrak{g}_x = \oplus_{i=1}^N \frac{\mathcal{F}^i}{\mathcal{F}^{i-1} + I_x \mathcal{F}^i}.$$

$l = (l_1, \cdots, l_N) \in \mathfrak{g}_x^*$. for every i, $l_1(X_1) = \lim_n t_n \xi_n(X_i(n))$.
For every i, j, $l_2([X_i, X_j]) = \lim_n t_n^2 \xi_n([X_i, X_j](x_n))$, etc..

Theorem (Helffer and Nourrigat 79)

The set $\mathcal{T}_x^*$ is closed under coadjoint orbit. So it corresponds by orbit method to a set of representations $\mathcal{T}_x$.

Main theorem

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D be of Hormander order l. Then TFAE

1. *for any $k \in \mathbb{N}$ and any u, $Du \in \tilde{H}^k(M)$ implies $u \in \tilde{H}^{k+l}(M)$.*
2. *for any $x \in M$, $\pi \in \mathcal{T}_x^* \subseteq \hat{G}_x$ (set of irreducible unitary representations), $\tilde{\sigma}(D, x, \pi)$ is injective.*
3. *(if M compact) for any $k \in \mathbb{N}$, $D : \tilde{H}^{k+l}(M) \to \tilde{H}^k(M)$ is left invertible modulo compact operators.*

Corollary

If M is compact Then TFAE

1. *D and D^* are maximally hypoelliptic*
2. *for any $k \in \mathbb{N}$, $D : \tilde{H}^{k+l}(M) \to \tilde{H}^k(M)$ is Fredholm.*

Index theory

$$D : C^\infty(M) \to C^\infty(M)$$

its index is

$$\mathrm{Ind}_a(D) = \dim(\ker(D)) - \mathrm{codim}(\mathrm{im}(D)).$$

Atiyah and Singer (1960) a topological formula for the index for elliptic operators.

D and D^* maximally hypoelliptic $\Rightarrow \dim(\ker(D))$ and $\mathrm{codim}(\mathrm{im}(D))$ are finite

Also $\mathrm{Ind}_a(D)$ is equal to the Fredholm index of $D : \tilde{H}^{k+l}(M) \to \tilde{H}^k(M)$ for any k.

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D and D^* maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\tilde{\sigma}(D))))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

- D and D^* maximally hypoelliptic implies $[\tilde{\sigma}(D)] \in K_0(C^*\mathcal{T})$.
- One has an isomorphism

  $$\mu : K_0(C^*\mathcal{T}) \to K^0(\mathcal{T}^*).$$

  The space $\mathcal{T}^* \subseteq \mathfrak{g}^*$ is the space that corresponds to $\mathcal{T}$ by orbit method.
- One has Excision map $Ex : K^0(\mathcal{T}^*) \to K^0(T^*M)$ and $\mathrm{Ind}_{AS} : K_0(T^*M) \to \mathbb{Z}$.

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k\partial_y$. Consider

$$D = (\partial_x^2 + (\sin(x)^k\partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth. Hormander order $k+1$. Our symbol of D at (x,y)

- if $\sin(x) \neq 0$, then $G_x = T_xM$ and our symbol=classical symbol
- if $\sin(x) = 0$. Then the Lie algebra of G_x is generated by

  $$\partial_x, x^k\partial_y, x^{k-1}\partial_y, \cdots, \partial_y$$

  with Lie bracket

  $$[\partial_x, x^j\partial_y] = jx^{j-1}\partial_y$$

  The two representations

  $$\begin{aligned} \pi_\pm : G_x &\to B(L^2\mathbb{R}) \\ \partial_x &\mapsto (f \mapsto \partial_t f) \\ x^j\partial_y &\mapsto (f \mapsto \pm it^j f) \end{aligned}$$

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D and D^* maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((\partial_t^2 - t^{2k})^{\frac{k+1}{2}})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number. Then

$$\begin{aligned}\mathrm{Ind}_a(D) = \quad & \textstyle\sum_\lambda w(g(0,y),\lambda) - w(g(0,y),-\lambda) \\ & + w(g(\pi,y),\lambda) - w(g(\pi,y),-\lambda)\end{aligned}$$

The End

Thank you for your attention