Equivariant analytic torsion for proper actions

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I Analytic torsion

The twisted exterior derivative

Let

- M be a compact, oriented Riemannian manifold of dimension n
- $\tilde{M}$ be the universal cover of M
- $\rho\colon \pi_1(M) \to \mathrm{U}(r)$ be a unitary representation of $\pi_1(M)$.

Then

$$\Omega^p_\rho(M) := (\Omega^p(\tilde{M}) \otimes \mathbb{C}^r)^{\pi_1(M)}$$

is the space of p-forms on M twisted by the flat vector bundle $\tilde{M} \times_{\pi_1(M)} \mathbb{C}^r$ defined by ρ.

We have the **twisted exterior derivaive**

$$d^p_\rho := d^p_{\tilde{M}} \otimes 1_{\mathbb{C}^r}\colon \Omega^p_\rho(M) \to \Omega^{p+1}_\rho(M).$$

It satisfies $d^p_\rho \circ d^{p-1}_\rho = 0$.

Twisted cohomology

Definition

The pth **de Rham cohomology of M twisted by ρ** is

$$H^p_\rho(M) := \ker(d^p_\rho)/\operatorname{im}(d^{p-1}_\rho).$$

Example

If ρ is the trivial representation, then

$$\Omega^p_\rho(M) = \Omega^p(M) \otimes \mathbb{C}$$
$$d^p_\rho = d^p_M$$
$$H^p_\rho(M) = H^p_{dR}(M) \otimes \mathbb{C}.$$

The Hodge Laplacian

Using the orientation and the Riemannian metric on M, we form the formal adjoint

$$(d_\rho^p)^* := (-1)^{np+n+1} * (d_\rho^{n-p}) *\colon \Omega_\rho^p(M) \to \Omega_\rho^{p-1}(M),$$

where $*$ is an extension of the Hodge $*$-operator to twisted forms.

Definition

The **Hodge Laplacian** on twisted p-forms is

$$\Delta_\rho^p := (d_\rho^{p+1})^* d_\rho^p + d_\rho^{p-1}(d_\rho^p)^*\colon \Omega_\rho^p(M) \to \Omega_\rho^p(M).$$

Theorem (Hodge)

The inclusion map $\ker(\Delta_\rho^p) \hookrightarrow \ker(d_\rho^p)$ *induces a linear isomorphism*

$$\ker(\Delta_\rho^p) \cong H_\rho^p(M).$$

Example: the circle

Let $M = S^1 = \mathbb{R}/\mathbb{Z}$. Let $\rho\colon \pi_1(S^1) = \mathbb{Z} \to \mathrm{U}(1)$ be given by

$$\rho(k) = e^{ik\alpha},$$

for an $\alpha \in \mathbb{R}$.

Then

$$\begin{aligned}
\Omega^0_\rho(S^1) &= \{f \in C^\infty(\mathbb{R}); f(x+1) = e^{i\alpha} f(x), \forall x \in \mathbb{R}\} \\
\Omega^1_\rho(S^1) &= \Omega^0_\rho(S^1)\, dx \\
\Delta^p_\rho &= -\frac{d^2}{dx^2}.
\end{aligned}$$

So

$$\begin{aligned}
H^p_\rho(S^1) &\cong \ker(\Delta^p_\rho) \\
&\cong \{f \in C^\infty(\mathbb{R}); f'' = 0,\ f(x+1) = e^{i\alpha} f(x), \forall x \in \mathbb{R}\} \\
&= \begin{cases} \mathbb{C} & \text{if } \alpha \in 2\pi\mathbb{Z} \\ \{0\} & \text{if } \alpha \notin 2\pi\mathbb{Z}. \end{cases}
\end{aligned}$$

So we get **no extra information** for nontrivial ρ. Idea: use vanishing of cohomology to define a **secondary invariant**, analytic torsion.

A ζ-function

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n/2$, set

$$\zeta_\rho^p(s) := \sum_{\lambda \in \mathrm{spec}(\Delta_\rho^p)\setminus\{0\}} \lambda^{-s} = \mathrm{Tr}\big((\Delta_\rho^p|_{\ker(\Delta_\rho^p)^\perp})^{-s}\big).$$

This converges by Weyl's law.

Theorem (Minakshisundaram–Pleijel, 1949)

The function ζ_ρ^p extends meromorphically to $\mathbb{C}$, and is regular near 0.

Analytic torsion

Definition (Ray–Singer, 1971)

The **analytic torsion** of M, twisted by ρ, is

$$T_\rho(M) := \exp\left(-\frac{1}{2}\sum_{p=1}^{n}(-1)^p p(\zeta_\rho^p)'(0)\right).$$

Theorem (Ray–Singer, 1971)

If $H_\rho^*(M) = \{0\}$, then $T_\rho(M)$ does not depend on the Riemannian metric used to define Δ_ρ^p.

Proof.

Let $(B_t)_{t\in[0,1]}$ be a smooth path of Riemannian metrics. Then

$$\frac{d}{dt}\log T_\rho(M; B_t) = \frac{1}{2}\sum_{p=1}^{n}(-1)^p \operatorname{tr}\left(\frac{d*_t}{dt} *_t^{-1}|_{\ker(\Delta_\rho^p)}\right).$$

Example: the circle

Consider the circle $M = S^1 = \mathbb{R}/L\mathbb{Z}$ of **circumference** L. The eigenfunctions of $\Delta_\rho^p = -d^2/dx^2$ on

$$\Omega_\rho^p(S^1) \cong \{f \in C^\infty(\mathbb{R}); f(x+L) = e^{i\alpha} f(x), \forall x \in \mathbb{R}\}$$

are

$$e_j(x) := e^{i(\alpha + 2\pi j)x/L},$$

for $j \in \mathbb{Z}$. So

$$\mathrm{spec}(\Delta_\rho^p) = \left\{ \left(\frac{\alpha + 2\pi j}{L} \right)^2 ; j \in \mathbb{Z} \right\}$$

(with multiplicities).
One then computes

$$T_\rho(\mathbb{R}/L\mathbb{Z}) = \begin{cases} 1/L & \text{if } \alpha \in 2\pi\mathbb{Z} \\ |2\sin(\alpha/2)|^{-1} & \text{if } \alpha \notin 2\pi\mathbb{Z}, \end{cases}$$

independent of L if and only if $H_\rho^*(\mathbb{R}/L\mathbb{Z}) = \{0\}$.

Regularised determinants

For a strictly positive definite matrix A, write

$$\zeta_A(s) := \mathrm{Tr}(A^{-s}).$$

Lemma

If A is a strictly positive definite matrix, then

$$\det(A) = e^{-\zeta_A'(0)}.$$

So it makes sense to define the **regularised determinant**

$$\det(\Delta_\rho^p|_{\ker(\Delta_\rho^p)^\perp}) := e^{-(\zeta_\rho^p)'(0)}.$$

Then

$$T_\rho(M) = \exp\left(-\frac{1}{2}\sum_{p=1}^{n}(-1)^p p(\zeta_\rho^p)'(0)\right) = \prod_{p=1}^{n} \det(\Delta_\rho^p|_{\ker(\Delta_\rho^p)^\perp})^{(-1)^p p/2}.$$

Reidemeister–Franz torsion

Ray and Singer defined analytic torsion as an an analytic way to compute **Reidemeister–Franz torsion**. This equals

$$\tau_\rho(M) := \prod_{p=1}^{n} \det(\widetilde{\Delta}_\rho^p|_{\ker(\widetilde{\Delta}_\rho^p)^\perp})^{(-1)^p p/2},$$

for a Laplace-type operator

$$\widetilde{\Delta}_\rho^p := (\tilde{d}_\rho^{p+1})^* \tilde{d}_\rho^p + \tilde{d}_\rho^{p-1}(\tilde{d}_\rho^p)^*,$$

where $\tilde{d}_\rho^p$ is a combinatorially defined boundary map in a finite-dimensional complex associated to a triangulation of M.

Theorem (Cheeger, Müller, late 1970s)

We have

$$T_\rho(M) = \tau_\rho(M).$$

There are generalisations by Bismut–Zhang (1992) and Müller (1993) to more general ρ.

II Equivariant analytic torsion

Equivariant analytic torsion

In the 1990s, various notions of equivariant analytic torsion were constructed. They applied to either

- actions by **finite** or **compact** groups; or
- actions by **fundamental groups** of compact manifolds on their universal covers.

Idea: replace the operator trace by (a special case of) the g-**trace**.

Proper actions

Let M now be a possibly noncompact, oriented Riemannian manifold.

Let G be a unimodular Lie group, acting on M, such that

- the action preserves the Riemannian metric and the orientation
- the action is **proper**, i.e. the map $G \times M \to M \times M$ given by

$$(g, m) \mapsto (m, gm)$$

  is proper
- M/G is compact.

Fix an element $g \in G$, with centraliser $Z := Z_G(g) < G$. Suppose that there is a G-invariant measure $d(xZ)$ on G/Z.

Examples

Example

If M and G are **compact**, and the action preserves the Riemannian metric and the orientation, then the conditions hold for all $g \in G$.

Example

If M is the universal cover of a compact, oriented, Riemannian manifold X, acted on by $G = \pi_1(X)$, then the conditions hold for all $g \in G$.

Example

If H is a reductive Lie group, $K < H$ is compact, and

- $M = H/K$
- $G < H$ is cocompact (e.g. $G = H$ or a cocompact lattice)
- $g \in G$ is semisimple (i.e. $\mathrm{Ad}(g)$ diagonalises),

then the conditions hold.

The g-trace

Because the action is proper, and M/G is compact, there is a **cutoff function** $\chi \in C_c^\infty(M)$ such that for all $m \in M$,

$$\int_G \chi(xm)\, dx = 1.$$

Suppose that $E \to M$ is a G-equivariant, Hermitian vector bundle. Let $T \in \mathcal{B}(L^2(E))^G$.

Definition

If $T\chi$ is trace class, and

$$\mathrm{Tr}_g(T) := \int_{G/Z} \mathrm{Tr}(xgx^{-1}T\chi)\, d(xZ)$$

converges, then this is the g-**trace** of T.

The g-trace does not depend on the choice of χ.

Examples

We had

$$\mathrm{Tr}_g(T) := \int_{G/Z} \mathrm{Tr}(xgx^{-1}T\chi)\, d(xZ).$$

Example

If $g = e$, then

$$\mathrm{Tr}_e(T) = \mathrm{Tr}(T\chi)$$

is the von Neumann trace of T.

Example

If M and G are compact, then we can take $\chi \equiv 1$. Then

$$\mathrm{Tr}_g(T) = \mathrm{vol}(G/Z)\,\mathrm{Tr}(g \circ T).$$

The ζ-function and heat operators

In the compact case,

$$\zeta_\rho^p(s) = \mathrm{Tr}\big((\Delta_\rho^p - P_\rho^p)^{-s}\big),$$

if $\mathrm{Re}(s) > n/2$, where P_ρ^p is projection onto $\ker(\Delta_\rho^p)$.

In our noncompact, equivariant setting, it is easier to work with **heat operators** than with negative powers of Laplacians.

Lemma

If $\mathrm{Re}(s) > n/2$, *then*

$$\zeta_\rho^p(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{Tr}(e^{-t\Delta_\rho^p} - P_\rho^p)\, dt.$$

Note:

- the integral converges near $t = 0$ because $\mathrm{Tr}(e^{-t\Delta_\rho^p}) \sim t^{-n/2}$
- the integral converges as $t \to \infty$ for any $s \in \mathbb{C}$, because $\mathrm{Tr}(e^{-t\Delta_\rho^p} - P_\rho^p)$ decays rapidly by Weyl's law.

A Hodge Laplacian

Let $F \to M$ be a G-equivariant, Hermitian, flat vector bundle. Let ∇^F be a G-invariant, flat connection on F preserving the metric. It extends to an operator

$$\nabla^F \colon \Omega^p(M; F) \to \Omega^{p+1}(M; F).$$

This has a formal adjoint

$$(\nabla^F)^* = (-1)^{np+n+1} * \nabla^F * \colon \Omega^p(M; F) \to \Omega^{p-1}(M; F).$$

Definition

The **Hodge Laplacian** on $\Omega^p(M; F)$ associated to ∇^F is

$$\Delta_F^p := (\nabla^F)^* \nabla^F + \nabla^F (\nabla^F)^*.$$

In the compact, non-equivariant case, $F = \tilde{M} \times_{\pi_1(M)} \mathbb{C}^r$ and $\nabla^F = d_\rho$ are determined by ρ.

The g-trace of heat operators

Proposition (B.L. Wang–H. Wang, H.–H. Wang)

The g-trace of the heat operator $e^{-t\Delta_F^p}$ converges if either

- *G/Z is compact*
- *G is discrete and finitely generated, or*
- *G is semisimple and g is semisimple.*

In particular, $\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)$ converges if the projection P_F^p onto the L^2-kernel of Δ_F^p is zero.

An equivariant ζ-function

Suppose that

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt$$

converges for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large, extends meromorphically to $s \in \mathbb{C}$, and is regular near zero, and that

$$\int_1^\infty t^{-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt$$

converges. (Do not want t^{s-1} here, because the integrand may not decay fast enough.)

Definition

$$(\zeta_{F,g}^p)'(0) := \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt + \int_1^\infty t^{-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt.$$

Equivariant analytic torsion

Suppose that

$$(\zeta_{F,g}^p)'(0) := \frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt + \int_1^\infty t^{-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt.$$

is well-defined.

Definition

The **equivariant analytic torsion** of M with respect to ∇^F and g is

$$T_g(\nabla^F) := \exp\left(-\frac{1}{2}\sum_{p=1}^{n}(-1)^p p(\zeta_{F,g}^p)'(0)\right).$$

This depends on ∇^F in general, just like it depends on ρ in the classical case.

Special cases

- If M and G are **compact**, then this yields equivariant versions of analytic torsion studied by Bismut, Bunke, Deitmar, Köhler, Lott, Lück, Rothenberg between 1991 and 1999.
- If M is the **universal cover** of a compact manifold X, and $G = \pi_1(X)$, then
  - if $g = e$, $F = M \times \mathbb{C}$ and $\nabla^F = d$, then $T_e(d)$ is L^2-**analytic torsion** (Lott, Mathai, 1991)
  - if G/Z is finite, $F = M \times \mathbb{C}$ and $\nabla^F = d$, then $T_g(d)$ is **delocalised analytic torsion** (Lott, 1999).
- For general M and G, and $g = e$, the number $T_e(\nabla^F)$ was studied by Guangxiang Su in 2013.

In all these cases, G/Z is compact.

Cheeger–Müller theorems were obtained in some of these settings, by various authors.

Example: the circle

Let $M = \mathbb{R}/L\mathbb{Z}$ for $L > 0$, and $G = \mathbb{R}/\mathbb{Z}$ acting on M by rotations. Let $F = \mathbb{R} \times_\rho \mathbb{C}$ and $\nabla^F = d_\rho$.
Then

- if $\alpha \notin 2\pi\mathbb{Z}$, then

$$T_e(\nabla^F) = |2\sin(\alpha/2))|^{-1}$$

$$T_{r+\mathbb{Z}}(\nabla^F) = \exp\left(\frac{1}{2}\sum_{j\in\mathbb{Z}} \frac{1}{|j-r|} e^{-i\alpha(j-r)}\right)$$

  if $r \notin \mathbb{Z}$

- if $\alpha \in 2\pi\mathbb{Z}$, then

$$T_e(\nabla^F) = 1/L.$$

Example: the line

Let $M = \mathbb{R}$, with Riemannian metric $L^2 dx^2$ for $L > 0$. Let $G = \mathbb{R}$, acting on M by translations. Let $F = \mathbb{R} \times \mathbb{C}$, with the action

$$g \cdot (x, z) = (x + g, e^{i\alpha g} z)$$

for $g, x \in \mathbb{R}$ and $z \in \mathbb{C}$. Let $\nabla^F = d$.
Then

$$T_0(\nabla^F) = 1$$
$$T_g(\nabla^F) = \exp\left(\frac{e^{-i\alpha g}}{2|g|}\right)$$

for $g \neq 0$.

Example: hyperbolic 3-space

Let $G = \mathrm{SO}_0(3,1)$, acting on hyperbolic 3-space $M = G/\mathrm{SO}(3)$. Let $g \in \mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3)$ be counter-clockwise rotation over an angle $x \neq 0$. Let $F = M \times \mathbb{C}$ and $\nabla^F = d$. Then

$$T_g(\nabla^F) = \exp\left(\frac{-1}{8\sin(x/2)^2}\right).$$

(Proof with Bismut's orbital integral trace formula.)

III Convergence

Švarc–Milnor functions

Let (g) be the conjugacy class of g.

Definition

A **Švarc–Milnor function** is a proper function $l\colon (g) \to [0, \infty)$ such that

- for all $c > 0$,

$$\int_{G/Z} e^{-cl(xgx^{-1})^2}\, d(xZ)$$

  converges
- for all compact subsets $Y \subset M$, there are $a, b > 0$ such that for all $m \in Y$ and $x \in G$,

$$\mathrm{dist}_M(xgx^{-1}m, m) \geq al(xgx^{-1}) - b.$$

Usually a natural candidate for l is an invariant distance function to $e \in G$.

Examples

Example

If G/Z is **compact**, then the zero function is a Švarc–Milnor function.

Example

If G is **discrete** and finitely generated, then a word length function is a Švarc–Milnor function.

- The first condition holds because G has at most exponential growth.
- The second condition follows from the Švarc–Milnor lemma.

Example

If G is a connected, real **semisimple** Lie group, and g is semisimple, then $l(x) = \mathrm{dist}_G(x, e)$ is a Švarc–Milnor function.

- The first condition is a result by Harish-Chandra.
- The second condition follows from Abels' slice theorem.

Small t convergence

Proposition (H.–Saratchandran)

If there is a Švarc–Milnor function, then

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\,dt$$

converges if $\mathrm{Re}(s)$ *is large enough. Then this extends meromorphically to* $\mathbb{C}$ *and is regular near* $s = 0$.

Proof.

Classical arguments in a compact set in G/Z containing eZ. Heat kernel estimates involving the Švarc–Milnor function outside this set. □

Large t convergence: Novikov–Shubin numbers

Convergence of the term

$$\int_1^\infty t^{-1} \operatorname{Tr}_g(e^{-t\Delta_F^p} - P_F^p)\, dt$$

in the equivariant ζ-function is a more difficult problem than small t convergence. This is **not known** in general, even for $g = e$ in the fundamental group case.

A sufficient condition for convergence is positivity of the following quantities.

Definition

The pth **Novikov–Shubin** number for g is

$$\alpha_g^p := \sup\{\alpha > 0; \operatorname{Tr}_g(e^{-t\Delta_F^p} - P_F^p) = \mathcal{O}(t^{-\alpha}) \text{ as } t \to \infty\}.$$

Invariance

Theorem (Gromov–Shubin, 1991)

Suppose that M is the universal cover of a compact manifold X, and that $G = \pi_1(X)$. Then the numbers α_e^p are homotopy invariants of X.

Novikov and Shubin already proved independence of the Riemannian metric in the late 1980s; Lott proved independence of the smooth structure in 1992.

Conjecture

Suppose that M is the universal cover of a compact manifold X, and that $G = \pi_1(X)$. Then $\alpha_e^p > 0$ for all p.

Questions

Under what conditions are the numbers α_g^p smooth, topological or homotopy invariants? When are they positive?

Examples

Example

If M and G are compact, then $\alpha_g^p = \infty$ because $\mathrm{Tr}(g \circ (e^{-t\Delta_F^p} - P_F^p))$ decays faster than any power of t.

Lemma (H.–Saratchandran)

If G/Z is compact and $\alpha_e^p > 0$, then $\alpha_g^p > 0$.

Consider the case where $G = \mathrm{SO}_0(n,1)$, acting on **hyperbolic space** $M = G/\mathrm{SO}(n)$. Let $F = M \times \mathbb{C}$ and $\nabla^F = d$.

Then

- $\alpha_g^p \geq 1/2$ for all p and all **hyperbolic** $g \in G$ (Fried 1986; Mathai 1991; H.–Saratchandran 2022)
- if $n = 3$ and g is **regular elliptic**, then $\sum_p (-1)^p p \,\mathrm{Tr}_g(e^{-t\Delta_F^p} - P_F^p) = \mathcal{O}(t^{-1/2})$ (H.–Saratchandran 2022).

There is ongoing work by S. Shen, Y. Song and X. Tang based on Bismut's trace formula, for semisimple G, $M = G/K$, and $g = e$.

IV Properties of equivariant analytic torsion

Metric independence

An important property of a notion of analytic torsion is independence of the Riemannian metric. For earlier versions of equivariant analytic torsion, this was proved for **finite** and **compact** conjugacy classes.

For **noncompact** conjugacy classes there is an interplay between

- volume growth of the conjugacy class, or of G/Z, and
- large time decay behaviour of heat kernels on differential forms.

Not much is known about the second point, apart from some results by Coulhon–Q.S. Zhang (2007), Devyver (2014), Coulhon–Devyver–Sikora (2020).

Metric independence

Theorem (H.–Saratchandran, 2022)

Suppose that the L^2-kernel of Δ_F^p is trivial for all p. Under conditions on the volume growth of G/Z and the large time behaviour of the heat kernel of Δ_F^p, equivariant analytic torsion converges, and is independent of the Riemannian metric.

Special cases:

- G/Z compact and $\alpha_e^p > 0$ for all p
- M simply connected, Δ_F invertible and G/Z with polynomial volume growth or slow enough exponential growth
- metrics in the same path component of the space of G-invariant Riemannian metrics satisfying a positive curvature condition, for G/Z with slow enough polynomial growth.

It is a very short argument that equivariant analytic torsion is invariant under rescaling the metric by a constant if Δ_F^p has trivial kernel for all p.

Triviality in even dimensions and a product formula

Some classical properties of analytic torsion generalise to the equivariant case.

Proposition (H.–Saratchandran, 2022)

If $n = \dim(M)$ is even, and $T_g(\nabla^F)$ converges, then it equals 1.

Proposition (H.–Saratchandran, 2022)

Suppose that for $j = 1, 2$, we have objects M_j, G_j, g_j, F_j and ∇^{F_j} like M, G, g, F and ∇^F. If the L^2-kernel of $\Delta^p_{F_j}$ is trivial for all p and $j = 1, 2$, then

$$T_{(g_1,g_2)}(\nabla^{F_1 \boxtimes F_2}) = T_{g_1}(\nabla^{F_1})^{\chi_{g_2}(F_2)} T_{g_2}(\nabla^{F_2})^{\chi_{g_1}(F_1)},$$

where $\chi_{g_j}(F_j)$ is an equivariant version of the Euler characteristic.

(At most one of the factors on the right is different from 1.)

Future work

In preparation:

- define an equivariant version of the Ruelle dynamical ζ-function and motivate an equivariant Fried conjecture/question.

Future:

- investigate invariance and positivity of Novikov–Shubin numbers
- investigate an equivariant Cheeger–Müller theorem.

Thank you