Extra twisted connected sums and their ν-invariants

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Outline

- G_2-Geometry
  Intro, properties, questions
- The ν-invariant
  Differential topology, definition of ν, properties, first examples
- Extra twisted connected sums
  Construction, properties, problems
- Computation of the ν-invariant
  Computations with η-invariants, examples, questions

Consider parallel translation along a spherical triangle

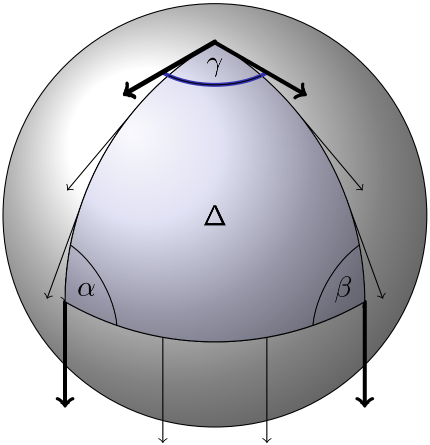
A vector is rotated by an angle equal to the spherical area of the triangle (Gauß-Bonnet theorem)

Parallel Translation generates

$$SO(2) \cong \mathrm{Hol}(S^2, g^{\mathrm{rnd}}) \subset \mathrm{Aut}(T_pS^2)$$

Related: the spherical area formula

$$A(\Delta) = \alpha + \beta + \gamma - \pi \,.$$



Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are

Holonomy group	dim	ric	*Structure*	*Parallel spinors*	*Name*
$SO(n)$	n				*generic case*
$U(k)$	$2k$		J		*Kähler*
$SU(k)$	$2k$	0	J, Ω	2	*Calabi-Yau*
$Sp(\ell) \cdot Sp(1)$	4ℓ	const	$\langle I, J, K \rangle$		*Quat. Kähler*
$Sp(\ell)$	4ℓ	0	I, J, K, Ω	$\ell + 1$	*hyper Kähler*
G_2	7	0	$\varphi \in \Omega^3$	1	*exceptional*
Spin(7)	8	0	$\psi \in \Omega^4$	1	*exceptional*

G_2-Geometry—Some motivation

Mathematical motivation

- Only special holonomy group for odd dimensional manifolds
- Only G_2 and $\mathrm{Spin}(7)$ holonomy have no direct relation to algebraic geometry

Hence, new methods are needed

Physical motivation

- In string theory, spacetime takes the form $\mathbb{R}^{3,1} \times V$, where V is Calabi-Yau
- In M-theory, spacetime takes the form $\mathbb{R}^{3,1} \times M$, where M is a G_2-manifold
- Possible relations to other physical theories

Hence, many fruitful interactions possible

Lie group $G_2 \subset GL(7,\mathbb{R})$ (pointwise) $\leftrightsquigarrow$ G_2-manifolds (global)

- Let $\varphi_0 = \langle \cdot \times \cdot, \cdot \rangle \in \Lambda^3 \operatorname{Im} \mathbb{O} \cong \Lambda^3\mathbb{R}^7$
  The stabiliser of φ_0 in $GL(7,\mathbb{R})$ is $G_2 = \mathrm{Aut}(\mathbb{O})$
  The $GL(7,\mathbb{R})$-orbit of $\varphi_0 \in \Lambda^3\mathbb{R}^7$ is open (not dense)
  Forms in this orbit are called positive
- Positive 3-form φ on M $\rightsquigarrow$ G_2-structure and metric g_φ

$$d\varphi = d^*_{g_\varphi}\varphi = 0 \qquad \text{torsion free (nonlinear condition)}$$
$$\iff \varphi \text{ is parallel}$$
$$\iff \mathrm{Hol}(M, g_\varphi) \subset G_2$$

- G_2 is simply connected $\rightsquigarrow$ G_2-manifolds are spin
  The stabiliser of a nonzero spinor in $\mathrm{Spin}(7)$ is G_2
  A Riemannian 7-manifold (M,g) has $\mathrm{Hol}(M,g) \subset G_2$
  if and only if it is spin and there exists a nonzero parallel spinor

Let M be a compact oriented spin 7-manifold and define

$$\mathcal{X} = \{ \varphi \in \Omega^3(M) \mid \varphi \text{ is positive and torsion free } \}$$

Let $\mathcal{D} \subset \mathrm{Diff}(M)$ be the connected component of id_M. Consider

$$\mathcal{M} = \mathcal{X}/\mathcal{D}$$

Theorem (Joyce)

The space $\mathcal{M}$ is a manifold, and the map

$$\mathcal{M} \longrightarrow H^3(M;\mathbb{R}) \qquad \textit{with} \qquad [\varphi] \longmapsto [\varphi]$$

is a local diffeomorphism

Not much is known about the global structure of $\mathcal{M}$

G_2-Geometry—Some open questions

Some fun facts

- Only a few obstructions against G_2-holonomy are known [Joyce]
- Only a few compact examples are known [Joyce, Kovalev, ...]
  —only $\sim 10^8$ deformation families
- Known compact examples are close to the boundary of the moduli space

Important open problems / questions

- Find more invariants for / obstructions against G_2-metrics
- Construct G_2-metrics far away from the boundary of the moduli space
- How can families of G_2-metrics become singular?
  How far can one deform a given G_2-metric?
- Construct G_2-metrics with prescribed singularities
  Certain singularities allow massless chiral Fermions to appear in M-theory

We want to describe G_2-manifolds using differential topology

Definition

A G_2-structure on a seven-manifold M is a reduction of the $GL(7, \mathbb{R})$-frame bundle to a bundle with structure group G_2

Equivalent descriptions

- Positive three form φ on M
- Riemannian metric, spin structure, and a unit spinor (up to sign)

Idea. Use nowhere vanishing spinors to describe and distinguish G_2-structures

Let σ_0, σ_1 be two nowhere vanishing spinors. Extend to $\bar{\sigma} \in \Gamma(\mathcal{S}^+(M \times [0,1]))$
A generic $\bar{\sigma}$ will have nondegenerate isolated zeros because

$$\operatorname{rk} \mathcal{S}^+(M \times [0,1]) \quad = \quad 8 \quad = \quad \dim(M \times [0,1])$$

Orient $\mathcal{S}^+(M \times [0,1])$ and count with signs

$$\Delta\nu(M; \sigma_0, \sigma_1) = 2 \cdot \#\bar{\sigma}^{-1}(0) = 2 \cdot \sum_{p \in \bar{\sigma}^{-1}(0)} \operatorname{sign}(d_p \bar{\sigma})$$

Theorem (Crowley-Nordström)

Let $F \colon M \to M$ be a spin diffeomorphism, then

$$\Delta\nu(M; \sigma, F^*\sigma) \quad \in \quad 48\mathbb{Z}$$

Can we write $\Delta\nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \in \mathbb{Z}/48$?

Idea. If M is spin, then M is the spin boundary of some compact 8-manifold W
Extend σ to $\bar{\sigma} \in \Gamma(\mathcal{S}^+ W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on W—not well-defined yet!

- $\chi(W)$—Euler characteristic of W
- $\operatorname{sign}(W)$—signature of W

Definition (Crowley-Nordström)

Assume that $M = \partial W$ with W spin, compact. Define

$$\nu(M, \sigma) = \chi(W) - 3\operatorname{sign}(W) - 2\#\bar{\sigma}^{-1}(0) \mod 48$$

Theorem (Crowley-Nordström)

$$\Delta\nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \quad \in \mathbb{Z}/48$$

Problem. Given M, how to determine W with $M = \partial W$?

Idea. Use the APS-index theorem and Mathai-Quillen currents

- $\psi(\nabla^{\mathcal{S}M}, g^{SM})$—Mathai-Quillen form in $\Omega^\bullet(\mathcal{S}M)$
- D_M—spin Dirac operator on $\Gamma(\mathcal{S}M)$
- B_M—odd signature operator $*d \pm d*$ on $\Omega^{\mathrm{ev}}(M)$
- h—dimension of the kernel
- η—Atiyah-Patodi-Singer η-invariant

Theorem (Crowley-G-Nordström)

$$\nu(M,\sigma) = 2\int_M \sigma^*\psi(\nabla^{SM}, g^{SM}) - 24(\eta + h)(D_M) + 3\eta(B_M) \in \mathbb{Z}/48$$

Proof.
Use $$2e(\nabla^{\mathcal{S}^+W}) = e(\nabla) + 48\hat{A}(\nabla)^{[8]} - 3L(\nabla)^{[8]} \in \Omega^8(W)$$ □

In the case of G_2-holonomy, things simplify

- σ is parallel, so $\sigma^*\psi(\nabla^{SM}, g^{SM}) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
- $\eta(D_M) \in \mathbb{R}$ is smooth on the G_2-moduli space $\mathcal{M}$

Definition (Crowley-G-Nordström)

Let (M, g) be a compact manifold with $\mathrm{Hol}(M, g) \subset G_2$. Put

$$\bar{\nu}(M, g) = 3\eta(B_M) - 24\eta(D_M) \quad \in \mathbb{Z}$$

- $\nu(M, \sigma) \equiv \bar{\nu}(M, g) - 24(1 + b_1(M)) \mod 48$
- $\bar{\nu}(M, g)$ is locally constant on $\mathcal{M}$
- $\bar{\nu}(M, g) = 0$ if M admits an orientation reversing isometry

The ν-invariant—First examples and questions

What about the known examples by Joyce and Kovalev?

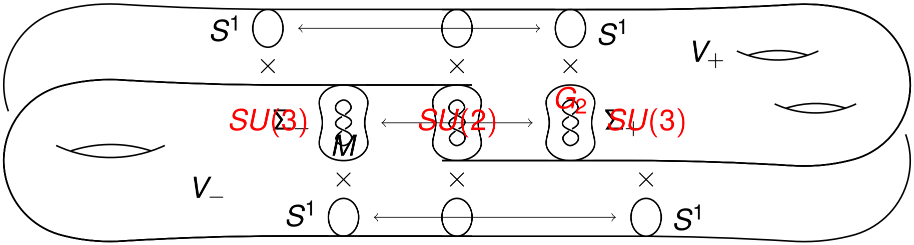
- $\bar{\nu}(M, g) = 0$ for all twisted connected sums
- $\bar{\nu}(M, g) = 0$ for many of Joyce's examples [Fornasin, PhD thesis]

Question. Is $\bar{\nu}(M, g) = 0$ whenever $\mathrm{Hol}(M, g) = G_2$?

- If yes, then $\bar{\nu}(M, g) \neq 0$ or $\nu(M, \sigma) \neq 24$ is a new obstruction against G_2-holonomy for a given G_2-structure
- If no, then $\bar{\nu}(M, g)$ is a non-trivial new invariant

Answer. We will construct examples with $\bar{\nu}(M, g) \neq 0$
(For Joyce's examples with $\nu \neq 0$, see [Scaduto, arXiv:2008.07239])
Using $\bar{\nu}(M, g)$, we can show that for some particular M,
the G_2-moduli space $\mathcal{M}$ has several connected components
—even with the same underlying G_2-structure

Twisted connected sums à la Kovalev and Corti-Haskins-Nordström-Pacini

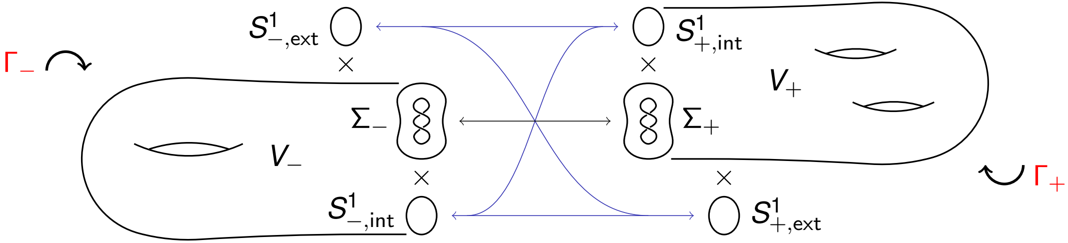


Let V_+, V_- be asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces

Glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles

There is a torsion-free G_2-structure close to the one obtained by gluing

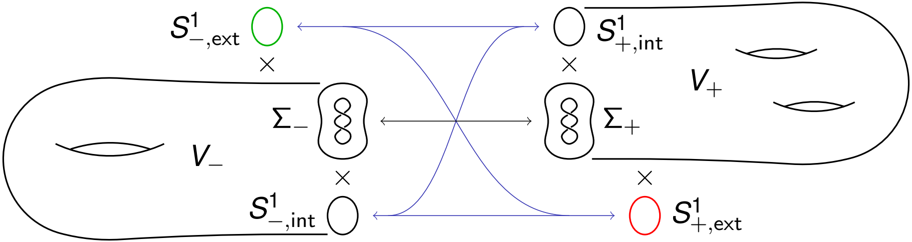
Extra twisted connected sums



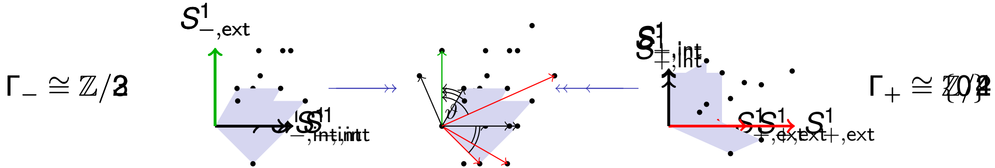
Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts both on $V_\pm$ and on $S^1_{\pm,\mathrm{ext}}$
The induced action on $\partial V_\pm$ has to fix $\Sigma_\pm$ pointwise
The actions on $S^1_{\pm,\mathrm{int}}$ and $S^1_{\pm,\mathrm{ext}}$ have to be free
Then $(S^1_{\pm,\mathrm{int}} \times S^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ is again a flat 2-torus
If both the tori and the K3 surfaces are isometric,
we can glue $M_\pm = (V_\pm \times S^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ at various angles ϑ

Extra twisted connected sums—Construction III

Extra twisted connected sums

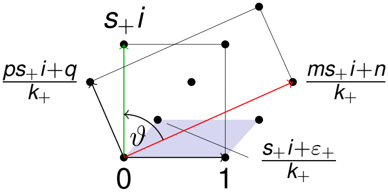


Modified gluing of tori at angle $\vartheta = \frac{3}{4}\pi \vartheta = \frac{2}{3}\pi \vartheta = \arccos\left(\frac{1}{\sqrt{6}}\right) \quad \notin \mathbb{Q}\pi$



Extra twisted connected sums—Matching the tori

Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts on $V_\pm \times S^1_{\pm,\mathrm{ext}}$
A torus matching is described by



- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$
- A gluing matrix $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$, here $\left(\begin{smallmatrix} 2 & 2 \\ 5 & -1 \end{smallmatrix}\right)$, with $\det G = -k_+ k_-$ and $mq \leq 0$, $np \geq 0$

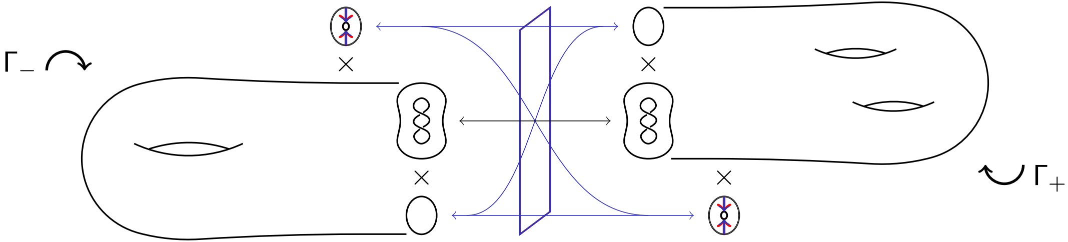
satisfying some extra conditions (only finitely many choices for ε_+, G possible)

From G, recover

- The aspect ratios $s_+ = \frac{\ell(S^1_{+,\mathrm{ext}})}{\ell(S^1_{+,\mathrm{int}})} = \sqrt{-\frac{nq}{mp}}$ and $s_- = \frac{\ell(S^1_{-,\mathrm{ext}})}{\ell(S^1_{-,\mathrm{int}})} = \sqrt{-\frac{mn}{pq}}$
- The gluing angle $\vartheta = \arg(ms_+ + in) \in (-\pi, \pi]$
- The fundamental group $\pi_1(M) \cong \mathbb{Z}/p$

There is a similar K3 matching problem involving a hyperkähler rotation by ϑ

Computation of the ν-invariant—Strategy



- Gluing formulas for η-Invariants [. . . , Bunke, Kirk-Lesch]
- Adiabatic limits (with boundary) [Bismut-Cheeger, Dai, G-Nordström]
- Variational formulas (with boundary) [Bismut-Cheeger, Dai-Freed]
- Analytic number theory [G-Nordström-Zagier]

Write $M = M_+ \cup_{M_0} M_-$ with $M_0 = \Sigma \times T^2$
Define Dirac operators D_{M_0} and B_{M_0} on M_0
There is a natural choice of Lagrangians $L_{B,\pm} \subset \ker(B_{M_0})$ and $L_{D,\pm} \subset \ker(D_{M_0})$

Using modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_\pm, g) = 3\eta_{\mathrm{APS}}(B_{M_\pm}; L_{B,\pm}) - 24\eta_{\mathrm{APS}}(D_{M_\pm}; L_{D,\pm}) \quad \in \quad \mathbb{R}$$

There exists $m(L_{B,+}, L_{B,-}) \in \mathbb{Z}$ depending only on the K3 matching such that

$$\bar{\nu}(M, g) = \bar{\nu}(M_+, g) + \bar{\nu}(M_-, g) + 144\frac{\vartheta}{\pi} - 72 + 3m(L_{B,+}, L_{B,-})$$

Recall that the gluing angle ϑ depends only on the gluing matrix

Compute adiabatic limit ($r \to 0$) of η-invariants on manifolds with boundary
We use the adiabatic limit theorems of [Bismut-Cheeger '92, Dai '01, G '14]

Except over the singular set of $V_\pm/\Gamma_\pm$,
the manifold $M_\pm$ is locally isometric to a product
The adiabatic limit depends only on isolated fixpoints of elements of $\Gamma_\pm$
Write

$$D_{\gamma_\pm}(V_\pm) = \lim_{r\to 0} \bar{\nu}(M_{\pm,r}) \quad \in \quad \mathbb{Q}$$

Computation of the ν-invariant—Variational formula

Shrink S^1-fibres of the flat orbibundle $M_\pm \to V_\pm/\Gamma_\pm$ to 0
By [Cheeger '87, Bismut-Cheeger '91, Dai-Freed '94]
the variational formula for the η-invariant consists of

- the integral of a Chern-Simons form over the interior
- the degree-1-component of an η-form on the boundary

The interior contribution vanishes because $M_\pm$ is locally a product

Let $\tilde\eta(\mathbb{A})$ be the η-form of the family of tori $(S^1_{\pm,\mathrm{int}} \times rS^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ for $r \in (0, s_\pm)$
The boundary contribution is given by

$$\bar\nu(M_\pm) - \lim_{r\to 0} \bar\nu(M_{\pm,r}) = F_{k_\pm,\varepsilon_\pm}(s_\pm) = 288 \int_0^{s_\pm} \tilde\eta(\mathbb{A})$$

Here, $s_\pm$ are the aspect ratios of the tori and depend only the gluing matrix

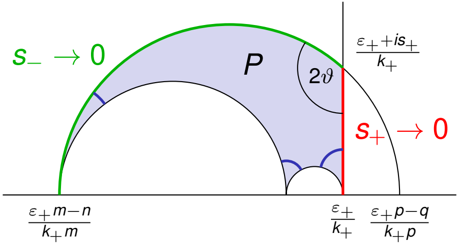
Represent the torus $(S^1_{\pm,\text{int}} \times rS^1_{\pm,\text{ext}})/\Gamma_\pm$ by $\tau \in \mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$
Then $\tilde\eta(\mathbb{A}) \in \Omega^1(\mathcal{H})$ is $SL(2,\mathbb{Z})$-invariant and satisfies $d\tilde\eta(\mathbb{A}) = -\frac{1}{4\pi}\, dA_{\text{hyp}}$
by the family index theorem [Bismut '86, Bismut-Cheeger '92]

Idea. Use hyperbolic geometry
to compute the η-form integrals

Adiabatic limits—geodesic rays

$\tilde\eta(\mathbb{A}) = 0$ along rectangular families
Determine P using continued fractions

Cusps—families of adiabatic limits
[Bunke-Ma '04, G-Nordström]



Use the hyperbolic area formula for P

The cusp contributions add up to a classical Dedekind sum [Zagier '75]
The angle 2ϑ at the finite corner cancels $144\frac{\vartheta}{\pi}$ in the gluing formula

An alternative approach uses Dedekind's η-function

The logarithm of the Dedekind η-function is given by

$$L(\tau) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty} \sum_{d|n} d^{-1}\, e^{2\pi i n \tau}$$

Let $\varepsilon_\pm^* \in \mathbb{Z}$ be such that $\varepsilon_\pm \varepsilon_\pm^* \equiv 1 \bmod k_\pm$. Then

$$\int_0^{s_\pm} \tilde{\eta}(\mathbb{A}) = -\frac{1}{\pi} \operatorname{Im} L\left(\frac{-\varepsilon_\pm^* + i s_\pm^{-1}}{k_\pm}\right) - \frac{\pi \varepsilon_\pm^*}{12 k_\pm}$$

Compute the sum of the variational terms using the functional equations

$$L(\tau + 1) = \frac{\pi i}{12} + L(\tau) \qquad \text{and} \qquad L\left(-\frac{1}{\tau}\right) = \frac{1}{2} \log \frac{\tau}{i} + L(\tau)$$

The correction terms add up to give a classical Dedekind sum

Computation of the ν-invariant—Final formula

Let M be an extra twisted connected sum
with gluing matrix $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ and gluing angle ϑ
let ε_+ and ε_+^* be as above with $\varepsilon_+\varepsilon_+^* \equiv 1 \bmod k_+$

For integers $n > 0$ and a define the classical Dedekind sum

$$S(a,n) = \sum_{j=1}^{n-1} \left(\left(\frac{j}{n}\right)\right)\left(\left(\frac{aj}{n}\right)\right) \in \frac{1}{6n}\mathbb{Z} \quad \text{with} \quad ((x)) = \begin{cases} 0 & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & x \notin \mathbb{Z} \end{cases}$$

Theorem (G-Nordström)

Assume $n > 0$. Then $a = \frac{m - \varepsilon_+^* n}{k_+} \in \mathbb{Z}$ and

$$\bar{\nu}(M,g) = D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) + 3m(L_{B_+}, L_{B_-}) + 24\left(\frac{q}{k_- n} - \frac{m}{k_+ n} + 12\,S(a,n)\right)$$

Example The example with $\cos\vartheta = \frac{1}{\sqrt{6}}$ has $\bar\nu(M,g) = -65$

Conjecture

All values in $\mathbb{Z}/48$ occur as ν-invariants of G_2-holonomy metrics

Questions

- How many different G_2-metrics exist on a given 7-manifold?
- Are there analogous invariants for singular G_2-spaces?

Example What is ν for an asymptotically cylindrical G_2-manifold like $M_\pm$?
Bad candidates: $\bar\nu(M_\pm, g_\pm) \in \mathbb{R}$
Better maybe: $D_{\gamma_+}(V_+) - 24\,\frac{\varepsilon_+^*}{k_+}$, $D_{\gamma_-}(V_-) - 24\,\frac{\varepsilon_-^*}{k_-} \in \mathbb{Z}$
But $\varepsilon_\pm^*$ is not well defined—and what does this mean?

Thanks for your attention!