Coherent sheaves, superconnection Riemann-Roch-Grothendieck

joint work J.-M. Bismut & Z. Wei (arXiv:2102.08129)

Shu Shen

IMJ-PRG, Sorbonne Université, Paris

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Main results

- X: closed complex manifold.
- $K(X)$: K-group of coherent sheaves.
  - holomorphic analogue of the topological K-theory.
- $H_{\mathrm{BC}}(X)$: Bott-Chern cohomology.
  - holomorphic analogue of the de Rham cohomology.

Theorem (Bismut-S.-Wei, 2021)

1. There is a Chern character $\mathrm{ch}_{\mathrm{BC}} : K(X) \to H_{\mathrm{BC}}(X)$.
2. $\mathrm{ch}_{\mathrm{BC}}$ satisfies a Riemann-Roch-Grothendieck formula for arbitrary holomorphic map $f : X \to Y$.

Contents

de Rham cohomology

- X: closed smooth manifold.
- $(\Omega^\bullet(X, \mathbf{R}), d)$: de Rham complex of smooth forms.
- $H^\bullet_{\mathrm{dR}}(X, \mathbf{R}) = \ker d / \mathrm{im} d$: de Rham cohomology.

Remark

We can define $H^\bullet_{\mathrm{dR}}(X, \mathbf{R})$ use the de Rham complex of currents $\mathcal{D}'(X, \Lambda^\bullet(T^*X))$.

Chern character

- D: smooth vector bundle on X.
- $\nabla^D : \Omega^\bullet(X,D) \to \Omega^{\bullet+1}(X,D)$: unitary connection.
- $R^D = \left(\nabla^D\right)^2 \in \Omega^2(X, \mathrm{End}(D))$: curvature.

Definition

$$\mathrm{ch}(D,\nabla^D) = \mathrm{Tr}\left[\exp(-R^D/2i\pi)\right].$$

Theorem (Chern-Weil)

1. $\mathrm{ch}(D,\nabla^D) \in \Omega^{\mathrm{even}}(X,\mathbf{R})$ *and is closed.*
2. $\mathrm{ch}(D) = [\mathrm{ch}(D,\nabla^D)] \in H^{\mathrm{even}}_{\mathrm{dR}}(X,\mathbf{R})$ *is independent of* ∇^D.

Bott-Chern cohomology

- X: closed complex manifold.
- $T_{\mathbf{C}}X = T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C} = TX \oplus \overline{TX}$: holomorphic and anti-holomorphic tangent bundles.
- $\Omega(X, \mathbf{C}) = \oplus_{p,q}\Omega^{p,q}(X, \mathbf{C})$.
- $d = \partial + \overline{\partial}$.

Definition (Bott-Chern)

$H^{p,q}_{\mathrm{BC}}(X, \mathbf{C}) = \ker d \cap \Omega^{p,q}(X, \mathbf{C})/\overline{\partial}\partial\Omega^{p-1,q-1}(X, \mathbf{C})$.

- We can define H_{BC} by currents.
- $H^{(=)}_{\mathrm{BC}}(X, \mathbf{R}) = \bigoplus_p H^{p,p}_{\mathrm{BC}}(X, \mathbf{R})$.

Bott-Chern vs de Rham

- Canonical morphism : $H^{p,q}_{\mathrm{BC}}(X,\mathbf{C}) \to H^{p+q}_{\mathrm{dR}}(X,\mathbf{C})$.
- If X is Kähler, $H_{\mathrm{BC}}(X,\mathbf{C}) \simeq H_{\mathrm{dR}}(X,\mathbf{C})$.
- In general, $H_{\mathrm{BC}}(X,\mathbf{C}) \neq H_{\mathrm{dR}}(X,\mathbf{C})$ (e.g. Iwasawa manifold).

Holomorphic vector bundles

- D: holomorphic vector bundle.
- $\nabla^{D''} : \Omega^{0,\bullet}(X, D) \to \Omega^{0,\bullet+1}(X, D)$ holomorphic structure.
  1. Leibniz rule: $\nabla^{D''}(\alpha s) = \overline{\partial}\alpha \cdot s + (-1)^{\deg \alpha} \alpha \wedge \nabla^{D''} s$
  2. $(\nabla^{D''})^2 = 0$.

Theorem (Koszul-Malgrange, Newlander-Nirenberg)

A smooth vector bundle D is holomorphic iff there is $\nabla^{D''} : \Omega^{0,\bullet}(X, D) \to \Omega^{0,\bullet+1}(X, D)$ with Leibniz rule and $(\nabla^{D''})^2 = 0$.

Chern connection

- $(D, \nabla^{D\prime\prime})$: holomorphic vector bundle.
- h^D : Hermitian metric on D.
- $\nabla^D = \nabla^{D\prime\prime} + \nabla^{D\prime}$: Chern connection. (Unique unitary connection whose antiholomorphic part is given by the holomorphic structure.)
- We have

$$\left(\nabla^{D\prime}\right)^2 = 0, \qquad \left(\nabla^{D\prime\prime}\right)^2 = 0, \qquad \left(\nabla^{D}\right)^2 = \left[\nabla^{D\prime\prime}, \nabla^{D\prime}\right].$$

- $R^D = \left(\nabla^D\right)^2 \in \Omega^{1,1}(X, \mathrm{End}(D))$.

Bott-Chern theory

- Chern-Weil: $\mathrm{ch}(D, h^D) = \mathrm{Tr}[\exp(-R^D/2i\pi)]$ closed real form.

Theorem (Bott-Chern)

1. $\mathrm{ch}(D, h^D) \in \Omega^{(=)}(X, \mathbf{R})$.
2. $\mathrm{ch}_{\mathrm{BC}}(D, \nabla^{D\prime\prime}) = [\mathrm{ch}(D, h^D)] \in H^{(=)}_{\mathrm{BC}}(X, \mathbf{R})$ *is independent of* h^D.

Complex of holomorphic vector bundles

- Complex of holomorphic vector bundles

$$0 \longrightarrow D^r \xrightarrow{v} D^{r+1} \xrightarrow{v} \ldots \longrightarrow D^{r'} \xrightarrow{v} 0 .$$

  - D^i has a holomorphic structure $\nabla^{D^i \prime\prime}$.
  - v is holomorphic, i.e., $\left[v, \nabla^{D\prime\prime}\right] = 0$.
- $A'' = v + \nabla^{D\prime\prime} : \Omega^{0,\bullet}(X, D^\bullet) \to \Omega^{0,\bullet}(X, D^\bullet)$ has total degree 1 and $(A'')^2 = 0$.
- A'' is an example of antiholomorphic superconnection.

$\mathrm{ch}_{\mathrm{BC}}$ for $(D^\bullet, v)$

- h^D : $\mathbf{Z}$-graded Hermitian metric on $D^\bullet$.
- $A' = v^* + \nabla^{D\prime}$.
- $A = A'' + A'$ (example of superconnection).
- $(A')^2 = 0, (A'')^2 = 0, A^2 = [A'', A']$.
- $\mathrm{ch}(D, A'', h^D) = \frac{1}{(2i\pi)^N}\mathrm{Tr_s}\left[\exp(-A^2)\right].$

Theorem (Bismut-Gillet-Soulé)

1. $\mathrm{ch}(D, A'', h^E) \in \Omega^{(=)}(X, \mathbf{R})$ *is d-closed.*
2. $\mathrm{ch}_{\mathrm{BC}}(D, A'') = [\mathrm{ch}(D, A'', h^D)] \in H^{(=)}_{\mathrm{BC}}(X, \mathbf{R})$ *is independent of* h^D.
3. $\mathrm{ch}_{\mathrm{BC}}(D, A'') = \sum_i (-1)^i \mathrm{ch}_{\mathrm{BC}}(D^i, \nabla^{D^i\prime\prime})$.

K-theory of holomorphic vector bundles

Definition

$K^\bullet(X)$: Abelian group

- *Generators : holomorphic vector bundles.*
- *Relations: if we have a short exact sequence,*

$$0 \to E \to E' \to E'' \to 0$$

then $E' = E + E''$.

Theorem

$\mathrm{ch}_{\mathrm{BC}} : K^\bullet(X) \to H_{\mathrm{BC}}^{(=)}(X, \mathbf{R})$.

Coherent sheaves

- The category of holomorphic vector bundles is not good.
- If $f : E \to F$ holomorphic bundle map, $\ker f$ and $\mathrm{im} f$ are not holomorphic vector bundles.
- If $(D^{\bullet}, v)$ is a complex of holomorphic vector bundles, denote $(\mathcal{O}_X(D^{\bullet}), v)$ the complex of sheaves associated to holomorphic sections.
- $\mathcal{H}(\mathcal{O}_X(D^{\bullet}), v)$ is an example of $\mathbf{Z}$-graged coherent sheaf.

Definition

$\mathscr{F}$ is a $\mathbf{Z}$-graded coherent $\mathcal{O}_X$-sheaf over X for any $x \in X$, there is $U \ni x$ such that $\mathscr{F}|_U$ can be obtained in the above way.

K-theory of coherent sheaves

Definition

$K(X)$: Abelian group

- *Generators : $\mathcal{E}$ coherent sheaves.*
- *Relations: if we have a short exact sequence*

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$$

then $\mathcal{E}' = \mathcal{E} + \mathcal{E}''$.

- $K^{\bullet}(X) \subset K(X)$.
- If X is projective, $K^{\bullet}(X) = K(X)$.
- In general, $K^{\bullet}(X) \neq K(X)$ (Voisin: a generic torus of dimension ≥ 3).

Questions

1. Is there a Chern Character $\mathrm{ch}_{\mathrm{BC}} : K(X) \to H^{(=)}_{\mathrm{BC}}(X, \mathbf{R})$?
2. Is $\mathrm{ch}_{\mathrm{BC}}$ compatible with the direct image associated to $f : X \to Y$ (RRG)?

Bismut-S.-Wei 2021: yes !

Remark

- Grothendieck 1956 : X, Y are projective and ch with taking in Chow groups.
- Atiyah-Hirzebruch 1962 : ch taking values in H_{dR} and RRG for immersion.
- O'Brian, Toledo, Tong, Levy…1980∼1990.
- Grivaux (2010) : ch taking values in $H_{\mathrm{Deligne}}(X, \mathbf{Q})$ and RRG for projective morphism.
- Wu (2020) : ch taking values in $H_{\mathrm{BC}}(X, \mathbf{Q})$ and RRG for projective morphism.
- Grivaux's unicity theorem: all the constructions of Chern Character are compatible.

Antiholomorphic superconnections

- $D^\bullet$: smooth $\mathbf{Z}$-graded vector bundles.

Definition (Quillen 85, Block 2010)

A differential operator $A'' : \Omega^{0,\bullet}(X, D^\bullet) \to \Omega^{0,\bullet}(X, D^\bullet)$ of total degree 1 is called an antiholomorphic superconnection,

1. If $\alpha \in \Omega^{0,p}(X)$, $s \in \Omega^{0,\bullet}(X, D)$,

$$A''(\alpha s) = \left(\overline{\partial}\alpha\right) \cdot s + (-1)^p \alpha \cdot A'' s$$

2. $(A'')^2 = 0$.

Smooth vs holo. complex of vector bundles

Write

$$A'' = v_0 + \nabla^{D''} + v_2 + \dots,$$

where $v_i \in \Omega^{0,i}(X, \mathrm{End}^{1-i}(D))$, $\nabla^{D''}$ antiholomorphic connextion s.t.

$$v_0^2 = 0, \qquad [\nabla^{D''}, v_0] = 0, \qquad (\nabla^{D''})^2 + [v_0, v_2] = 0, \qquad \dots$$

Example

If $v_2 = v_3 = \dots = 0$, then

$$v_0^2 = 0, \qquad [\nabla^{D''}, v_0] = 0, \qquad (\nabla^{D''})^2 = 0.$$

By Koszul-Malgrange/Newlander-Nirenberg, (D, v_0) is a complex of holomorphic vector bundles.

Block's Theorem

1. Given (D, A''), we can define a complex of sheaves $\mathscr{E}$, such that

$$\text{if } U \subset X, \mathcal{E}(U) = \left(\Omega^{0,\bullet}(U, D|_U), A''|_U\right).$$

2. $\mathscr{H}\mathscr{E}$: cohomology of $\mathscr{E}$.

Theorem (Block 2010, Bismut-S.-Wei 2021)

1. $\mathscr{H}\mathscr{E}$ is a $\mathbf{Z}$-graded coherent sheaf.
2. Every $\mathbf{Z}$-graded coherent sheaf can be obtained in this way.

Proof.

1. Locally, after conjugaison, $A'' \simeq v + \nabla''$ (extension of Koszul-Malgrange/Newlander-Nirenberg).
2. $(D, A'') \to \mathscr{H}\mathscr{E}$ defines an equivalence of categories.

□

Chern-Weil theory for (D, A'')

- h: $\mathbf{Z}$-graded Hermitian metric on $D^\bullet$.
- $A = A'' + A'$: unitary superconnection.
- $(A')^2 = 0, (A'')^2 = 0$ and $A^2 = [A', A'']$.

Definition

$\mathrm{ch}(D, A'', h) = \frac{1}{(2i\pi)^{N/2}} \mathrm{Tr_s}[\exp(-A^2)]$.

Theorem (Bismut-S.-Wei 2021)

1. $\mathrm{ch}(D, A'', h) \in \Omega^{(=)}(X, \mathbf{R})$ and d-closed.
2. $\mathrm{ch_{BC}}(D, A'') = [\mathrm{ch}(D, A'', h)]$ in $H^{(=)}_{\mathrm{BC}}(X, \mathbf{R})$ is independent of h.
3. $\mathrm{ch_{BC}}$ descends to $K(X) \to H^{(=)}_{\mathrm{BC}}(X, \mathbf{R})$.

Statement of RRG

- X, Y: closed complex manifolds.
- $f : X \to Y$: holomorphic map
- Grauert : if $\mathcal{E}$ is coherent on X, then $f_*\mathcal{E}$ is still coherent on Y.
- $f_! : K(X) \to K(Y)$.
- $H_{\mathrm{BC}}^{(=)}(X, \mathbf{R})$ can be defined by currents.
- $f_* : H_{\mathrm{BC}}^{(=)}(X, \mathbf{R}) \to H_{\mathrm{BC}}^{(=)}(Y, \mathbf{R})$.

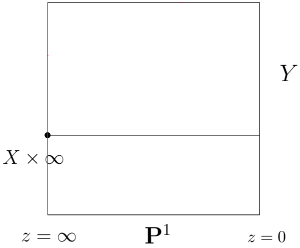
Theorem (Bismut-S.-Wei 2021)

If $\mathscr{F} \in K(X)$, then

$$\mathrm{Td}_{\mathrm{BC}}(TY)\mathrm{ch}_{\mathrm{BC}}(f_!\mathscr{F}) = f_* \left(\mathrm{Td}_{\mathrm{BC}}(TX)\mathrm{ch}_{\mathrm{BC}}(\mathscr{F})\right) \ \textit{in } H_{\mathrm{BC}}^{(=)}(Y, \mathbf{R}).$$

RRG for immersions : deformation to normal cone

$$W = \mathrm{Bl}_{X\times\infty}(Y \times \mathbf{P}^1).$$



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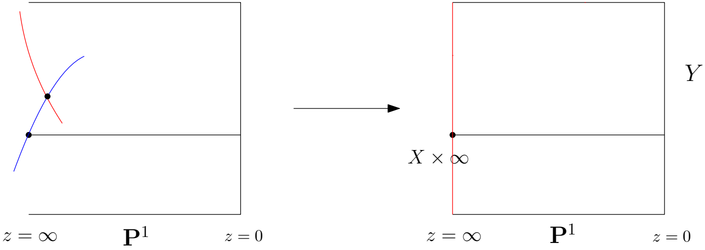

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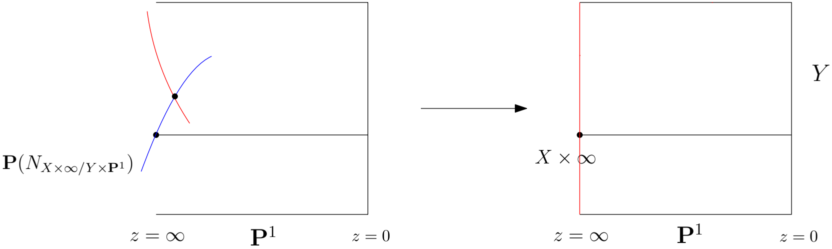

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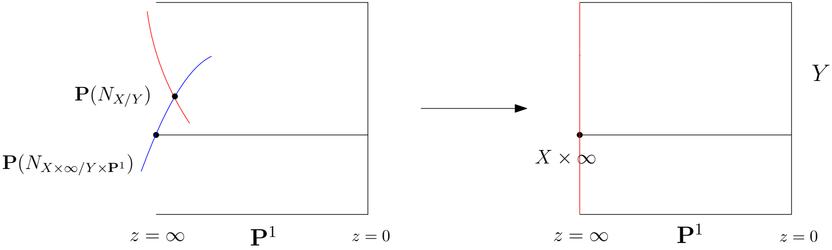

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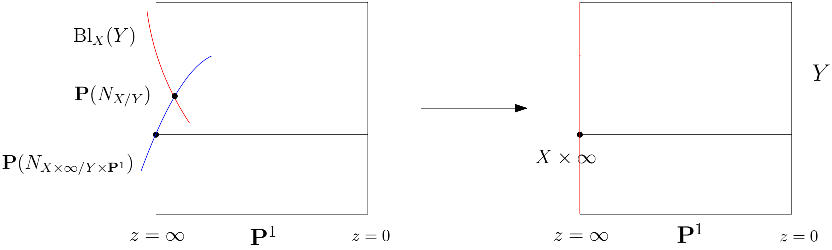

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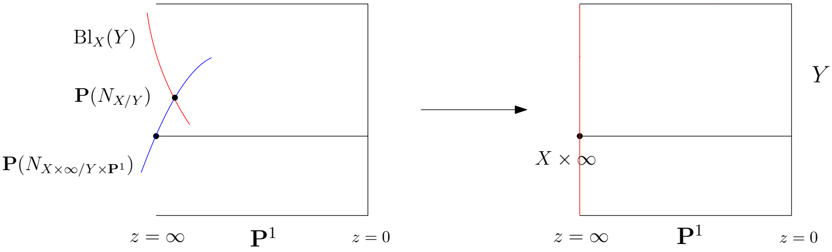


Deform an immersion $X \to Y$ to an other immersion

$$X \to \mathbf{P}(N_{X\times\infty/Y\times\mathbf{P}^1}).$$

Direct image for projection

- It is enough to consider $\pi : M = X \times S \to S$.
- We assume $\mathscr{F} \in K(M)$ is defined by $(D^\bullet, A'')$ over M.
- $\mathcal{D}^\bullet = \Omega^{0,\bullet}(X, D^\bullet|_X)$: infinite dimensional $\mathbf{Z}$-graded vector bundle on S.
- $\Omega^{0,\bullet}(S, \mathcal{D}^\bullet) = \Omega^{0,\bullet}(M, D^\bullet)$
- an antiholomorphic superconnection $\mathcal{A}'' = A''$.
- $\pi_! \mathscr{F}$ is "represented" by $(\mathcal{D}, \mathcal{A}'')$.

Elliptic Chern character

- Given metrics g^D and g^{TX}, we can define an L^2-metric on $\mathcal{D} = \Omega^{0,\bullet}(M, D^\bullet)$.
- $\mathcal{A} = \mathcal{A}' + \mathcal{A}''$ fibrewise elliptic.
- $\mathrm{ch}(\mathcal{D}, \mathcal{A}'', g^D, g^{TX}) = \frac{1}{(2i\pi)^{N/2}} \mathrm{Tr_s}[\exp(-\mathcal{A}^2)]$.

Theorem (Bismut-S.-Wei 2021)

1. $\mathrm{ch}(\mathcal{D}, \mathcal{A}'', g^D, g^{TX}) \in \Omega^{(=)}(S, \mathbf{R})$ and d-closed.
2. Its class $\mathrm{ch_{BC}}(\mathcal{D}, \mathcal{A}'')$ in $H^{(=)}_{\mathrm{BC}}(S, \mathbf{R})$ is independent of g^D, g^{TX}, and

$$\mathrm{ch_{BC}}(\mathcal{D}, \mathcal{A}'') = \mathrm{ch_{BC}}(\pi_! \mathcal{F}).$$

Proof.

spectral troncation + fibrewise Hodge theory.

Byproduct: a new proof of Grauert's theorem.

Atiyah-Singer index theorem

- $S = * :$ by Atiyah-Singer,

$$\mathrm{ch}_{\mathrm{BC}}(\pi_! F) = \mathrm{ch}_{\mathrm{BC}}(\mathcal{D}, \mathcal{A}'') = \mathrm{ind}(\mathcal{A}_+) \\ = \int_X \mathrm{Td}_{\mathrm{BC}}(TX)\mathrm{ch}_{\mathrm{BC}}(\mathrm{E}).$$

- S general : family index theory of Atiyah-Singer implies RRG in H_{dR}.
- To get RRG in H_{BC}, we need the local family index theorem.

Elliptic local index theorem

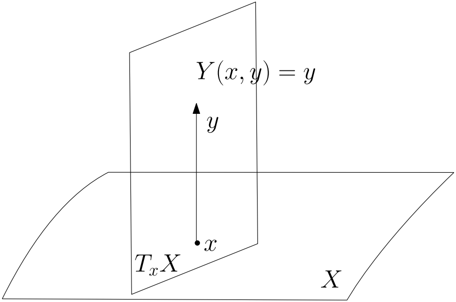
- J^{TX} complex structure on X, $\omega^X = g^{TX}(\cdot, J^{TX}\cdot)$.
- If $\overline{\partial}^X \partial^X \omega^X = 0$, by local family index theorem, as $t \to 0$,

$$\mathrm{ch}(\mathcal{D}, \mathcal{A}'', g^D, g^{TX}/t) \to \text{some limit in } \Omega^{(=)}(S, \mathbf{R})$$
$$\equiv \pi_* \left(\mathrm{Td}_{\mathrm{BC}}(TX)\mathrm{ch}_{\mathrm{BC}}(E)\right) \text{ in } H^{(=)}_{\mathrm{BC}}(S, \mathbf{R}).$$

- If $\overline{\partial}^X \partial^X \omega^X \neq 0$, there are some divergence terms after Getzler's rescaling.

Dolbeault-Koszul resolution

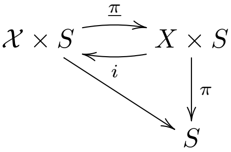
- $\mathcal{X} = TX$. $i : X \to \mathcal{X}$ by zero section. $Y \in C^\infty(\mathcal{X}, \pi^*TX)$.



- Dolbeault-Koszuel: $i_! \mathcal{O}_X$ is represented by

$$\left(\pi^*\Lambda^\bullet(T^*X), \overline{\partial}^{\mathcal{X}} + i_Y\right).$$

Enlarge the fibration



- $i_! \mathscr{F}$ and $\mathscr{F}$ are expected to have the same direct image on S.
- $i_! \mathscr{F}$ is associated to the antiholomorphic superconnection

$$\left(\underline{\pi}^* \left(\Lambda(T^*X)\widehat{\otimes}D\right), \underline{\pi}^* A'' + i_Y\right).$$

Hypoelliptic deformation

- Infinite dimensional obj on S:

$$\left(\underbrace{\Omega^{0,\bullet}(\mathcal{X}, \underline{\pi}^*\left(\Lambda(T^*X)\widehat{\otimes}D\right)}_{\mathcal{D}}, \mathcal{A}''_Y\right)$$

- $\mathcal{D} = \Omega^{\bullet,\bullet}(X, \Omega^{0,\bullet}(TX) \otimes D)$.
- $g^D, g^{TX} \rightsquigarrow L^2$-metric on $\Omega^{0,\bullet}(TX) \otimes D$.
- $\omega^X \rightsquigarrow$ non degenerate Hermitian form $\left(\frac{i}{2\pi}\right)^{\dim X} \int_X \widetilde{\alpha} \wedge \overline{e^{-i\omega^X}\beta}$
- We get non degenerate Hermitian form on $\mathcal{D}$.
- $\mathcal{A}_Y = \mathcal{A}''_Y + \mathcal{A}'_Y$, $\mathcal{A}^2_Y$ is hypoelliptic,

$$\mathcal{A}^2_Y = \frac{1}{2}(-\Delta^V + |Y|^2 + \ldots) + \nabla_{Y^H} + \ldots$$

Hypoelliptic Chern-Weil theory

- We can define $\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X)$ as before.

Theorem

1. $\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X) \in \Omega^{(=)}(S, \mathbf{R})$ *and d-closed*
2. $[\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X)] \in H^{(=)}_{\mathrm{BC}}(S, \mathbf{R})$ *is independent of* g^D, g^{TX}, ω^X.
3. $[\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X)] = \mathrm{ch}_{\mathrm{BC}}(\pi_! \mathscr{F}) \in H^{(=)}_{\mathrm{BC}}(S, \mathbf{R})$.

Proof.

Part 3 is based on the fact that the hypoelliptic curvature $\mathcal{A}^2_Y$ can be deformed to the elliptic curvature $\mathcal{A}^2$. As $b \to 0$, we have (Bismut-Lebeau 08)

$$\mathrm{ch}(\mathcal{A}''_Y, g^D, b^4 g^{TX}, \omega^X) \to \mathrm{ch}(\mathcal{A}'', g^D, g^{TX}) \text{ in } \Omega(S, \mathbf{R}).$$

□

Hypoelliptic local index theorem

- If $\overline{\partial}^X \partial^X \omega^X = 0$, as $t \to 0$,

$$\mathrm{ch}\left(\mathcal{A}''_Y, g^D, g^{TX}/t^3, \omega^X/t\right) \to \pi_*\left(\mathrm{Td}(TX, g^{TX})\mathrm{ch}(D, A'', g^D)\right)$$
$$\text{in } \Omega^{(=)}(S, \mathbf{R}),$$

- If we replace ω^X by $|Y|^2\omega^X$ in the construction,

$$\mathrm{ch}(\dots) \to \pi_*\left(\mathrm{Td}(TX, g^{TX})\mathrm{ch}(E, A'', g^D)\right)$$

  without any assumptions!
- The fact that $|Y|^2\omega^X|_X = 0$ kills the divergence from Getzler's transformations.

References

J.-M. Bismut, S. Shen, and Z. Wei. Coherent sheaves, superconnections, and RRG. arXiv:2102.08129, 2021.

J.-M. Bismut. *Hypoelliptic Laplacian and Bott-Chern cohomology*, *Progress in Mathematics* 305. Birkhäuser, 2013.

J. Block. Duality and equivalence of module categories in noncommutative geometry. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 311–339. Amer. Math. Soc., Providence, RI, 2010.

Thank you for your attention !
Happy birthday, Paolo!