Positive Scalar Curvature from a Concordance Viewpoint

Dirac Operators in Topology, Geometry, and Representation Theory

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GEOMETRY AT INFINITY

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Motivation

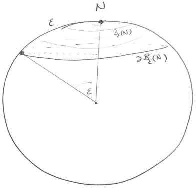
Scalar Curvature

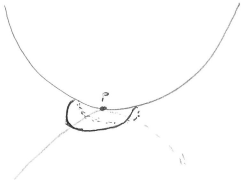
Definition

Scalar curvature $\mathrm{scal}_g \colon M^d \to \mathbb{R}$ measures the asymptotical growth of (geodesic) balls compared to euclidean space:

$$\frac{\mathrm{vol}(B_\varepsilon(p) \subseteq M^d)}{\mathrm{vol}(B_\varepsilon(0) \subseteq \mathbb{R}^d)} = 1 - \frac{\mathrm{scal}_g(p)}{6(d+2)}\varepsilon^2 + \mathcal{O}(\varepsilon^4)$$

$$\frac{\mathrm{Area}(\partial B_\varepsilon(p) \subseteq M^d)}{\mathrm{Area}(\partial B_\varepsilon(0) \subseteq \mathbb{R}^d)} = 1 - \frac{\mathrm{scal}_g(p)}{6d}\varepsilon^2 + \mathcal{O}(\varepsilon^4)$$





How many Riemannian Metrics does M have?

$R^+(M) := \{g \text{ Riemannian metric on } M,\ \mathrm{scal}(g) > 0\}$ with smooth Fréchet topology.

Question

- What is the homotopy type of $R^+(M)$?
- How rich/complicated are the homotopy groups of $R^+(M)$?

Relation on PSC Space

Definition

Two Riemannian metrics $g_0, g_1 \in R^+(M)$ are called isotopic, if there is a smooth path $(g_t)_{t\in[0,1]}$ joining them.

Definition

Two Riemannian metrics $g_0, g_1 \in R^+(M)$ are called concordant, if there is a $\widetilde{g} \in R^+(M \times [0,1])$ that decomposes into $g_j + \mathrm{d}t^2$ near the boundary. The metric $\tilde{g}$ is called a concordance between g_0 and g_1.

Today's Goal

Theorem (H.)

The following diagram commutes:

$$\begin{array}{ccc} R^+(M^d) & \xrightarrow{\mathrm{inddif}_H(\cdot,g_0)} & KO^{-(d+1)} \\ \big\downarrow & & \big\downarrow \simeq \\ \widetilde{R}^+(M^d) & \xrightarrow{\mathrm{inddif}_{GL}(\cdot,g_0)} & KO^{-(d+1)} \end{array}$$

Cubical Set Theory

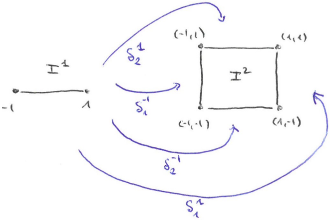
Singular Set

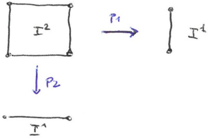
Example

The singular set $S_\bullet(X)$ is given by

$$S_n(X) = \{f\colon I^n \to X \text{ continuous }\},$$
$$\partial_i^\varepsilon(f) = f \circ \delta_i^\varepsilon, \qquad \sigma_i(f) = f \circ p_i,$$

where $\delta_i^\varepsilon\colon I^{n-1} \to I^n$ includes ε into the i-th component, and $p_i\colon I^{n+1} \to I^n$ projects the i-th component away.





Definition of Cubical Sets

Definition

A cubical set is a sequence of sets $(X_n)_{n\geq 0}$ together with connecting maps $\partial_i^\varepsilon \colon X_n \to X_{n-1}$ and $\sigma_j \colon X_n \to X_{n+1}$, where $1 \leq i \leq n$, $\varepsilon \in \mathbb{Z}_2$, and $1 \leq j \leq n+1$, that satisfy certain cubical identities.

Definition

A cubical map is a sequence of maps $f_n \colon X_n \to Y_n$ that commute with the connecting maps.

Homotopy-theoretical Interplay

Theorem

The functors

$$|\cdot|: \mathbf{cSet} \rightleftarrows \mathbf{CW} : S_\bullet(\cdot)$$

form an adjoint pair.
If $X_\bullet, Y_\bullet$ are good cubical sets, then they induce a bijection on homotopy classes

$$|\cdot|: [X_\bullet, Y_\bullet] \underset{}{\overset{1:1}{\rightleftarrows}} [X, Y] \; : S_\bullet(\cdot)$$

The Concordance Set and its Friends

Combinatorial Version of the PSC Space

Definition

A map $\varphi\colon \mathbb{R}^n \to X$ is a block map if there is an $R > 0$ such that

$$\varphi(x_1, \dots, x_n) = \varphi(x_1, \dots, \pm R, \dots, x_n) \qquad \text{if } \pm x_i \geq R$$

for all $1 \leq i \leq n$.

Definition

Let $R^+_\bullet(M)$ be the cubical given by

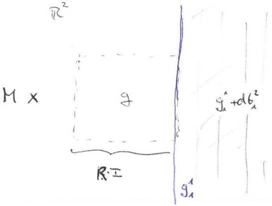
$$R^+_n(M) = \{g\colon \mathbb{R}^n \to R^+(M) \ :\ g \text{ smooth block map } \}$$
$$\partial_i^\varepsilon(g) = \lim_{R\to\infty} g \circ \delta_i^{R\varepsilon} \quad \text{and} \quad \sigma_i(g) = g \circ p_i.$$

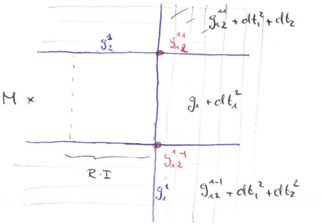
Block Metrics

Definition (block metrics)

A Riemannian metric g on $M \times \mathbb{R}^n$ is a block metric if there is a $R > 0$ such that

$$g|_{M\times\{\varepsilon x_i > R\}} = g \restriction_{M\times\{x_i = R\varepsilon\}} + \mathrm{d}x_i^2.$$





The Concordance Set

$M = M^d$ closed manifold of dimension d.

Definition

$\widetilde{R}^+_\bullet(M)$ is the cubical set given by

$$\begin{aligned} \widetilde{R}^+_n(M) &:= \{g \text{ block metric on } M \times \mathbb{R}^n, \operatorname{scal}(g) > 0\} \\ \partial_i^\varepsilon(g) &:= \lim_{R \to \infty} g \restriction_{M \times \{x_i = R\varepsilon\}} \\ \sigma_i(g) &:= p_i^* g + \mathrm{d}x_i^2. \end{aligned}$$

Remark: This definition makes sense without the psc condition. The resulting set is denoted by $\widetilde{R}_\bullet(M)$.

The Concordance Set is Good

Theorem (H.)

The concordance set $\widetilde{R}^+_\bullet(M)$ is a Kan set.

Example

- $\pi_0(\widetilde{R}^+_\bullet(M)) = \{\text{concordance classes of psc metrics}\}$.
- $\pi_1(\widetilde{R}^+_\bullet(M), g_0) = \{\text{concordance classes of self concordances of } g_0\}$
- $\pi_n(\widetilde{R}^+_\bullet(M), g_0) = \{g \in \widetilde{R}^+_n(M) \,:\, \partial^\omega_j g = g_0\}/\text{concordance}$

The PSC Suspension Map

Definition

For $g \in R_n^+(M)$, let $\mathrm{susp}_n(g)$ be the Riemannian metric on $M \times \mathbb{R}^n$ given by

$$\mathrm{susp}_n(g)(m,t) := g(t)_m + \sum_{j=1}^{n} \mathrm{d}t_j^2.$$

These maps assemble to a cubical map $\mathrm{susp}_\bullet \colon R_\bullet^+(M) \to \widetilde{R}_\bullet(M)$.

Theorem

There is a cubical subset $A_\bullet$ of $R_\bullet^+(M)$ such that

$$R_\bullet^+(M) \xleftarrow{\simeq} A_\bullet \xrightarrow{\mathrm{susp}_\bullet} \widetilde{R}_\bullet^+(M).$$

Digression into Index Theory

Setup

- (M^d, g) closed Riemannian spin manifold
- $\mathfrak{S}_g \to M$ its Spinor bundle
  - $\mathfrak{S}_g = \mathfrak{S}_g^+ \oplus \mathfrak{S}_g^-$ graded
  - $\mathfrak{S}_g \curvearrowleft Cl_{d,0}$ is Clifford right linear.
- $\not{\mathfrak{D}}_g : \Gamma(\mathfrak{S}_g) \to \Gamma(\mathfrak{S}_g)$ Dirac operator.

Connection To Positive Scalar Curvature

$$\not{\mathfrak{D}}_g^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}(g) \qquad \text{(BLSW-formula)}$$

$$[\ker \not{\mathfrak{D}}_g] = \alpha(M) \in KO^{-d}(pt) \qquad \text{(Atiyah-Singer)}$$

Pseudo Dirac Operators

Definition

$\Psi\mathrm{Dir}(M) \to \mathsf{Riem}(M)$ is the affine bundle given by

$$\Psi\mathrm{Dir}(M)_g = \{P \in \Psi\mathrm{DO}^1_{Cl}(\mathfrak{S}_g) \;:\; \text{s.a., odd, } \mathrm{symb}(P) = \mathrm{symb}(\not{\mathfrak{D}}_g)\}$$

Theorem (Ebert '17)

For M^d closed, $\Psi\mathrm{Dir}^\times(M^d)$ is a classifying space for real K-theory:

$$[X, \Psi\mathrm{Dir}^\times(M^d)] \cong KO^{-(d+1)}(X)$$

and the isomorphism is given by the index difference.

Cubical Version of Pseudo Dirac Operators

Definition

Let $\Psi\mathrm{Dir}^{\times}_{\bullet}(M)$ be the cubical set with

$$\Psi\mathrm{Dir}^{\times}_{n}(M) := \{P\colon \mathbb{R}^n \to \Psi\mathrm{Dir}^{\times}(M) \text{ smooth block map}\}$$

$$\partial_i^{\varepsilon}(P) := \lim_{R\to\infty} P \circ \delta_i^{R\varepsilon} \quad \text{ and } \quad \sigma_i(P) := P \circ p_i.$$

The indexdifference in this setup is now easily described by the cubical map

$$\not{\mathfrak{D}}_{\bullet}\colon R^{+}_{\bullet}(M) \to \Psi\mathrm{Dir}^{\times}_{\bullet}(M), \qquad g \mapsto \not{\mathfrak{D}}_{g}\,.$$

Review

$$
\begin{array}{ccc}
R^+(M) & \xrightarrow{\text{inddif}} & KO^{-(d+1)} \\
\big\downarrow & & \big\downarrow \\
\widetilde{R}^+(M) & \longrightarrow & KO^{-(d+1)}
\end{array}
\qquad\qquad
\begin{array}{ccc}
R^+_\bullet(M) & \xrightarrow{\mathfrak{D}_\bullet} & \Psi\mathrm{Dir}^\times_\bullet(M^d) \\
{\scriptstyle \mathrm{susp}_\bullet}\big\downarrow & & \\
\widetilde{R}^+_\bullet(M) & &
\end{array}
$$

Factorisation Theorem

Theorem (H.)

Let M^d be a closed spin manifold with psc metric. There are homotopy equivalent cubical subsets $A_\bullet \subseteq R^+_\bullet(M)$ and $B_\bullet \subseteq \Psi\mathrm{Dir}^\times_\bullet(M)$ such that the diagram

$$
\begin{array}{ccccc}
R^+_\bullet(M) & \xleftarrow[]{\;\simeq\;} & A_\bullet & \xrightarrow{\;\mathrm{susp}_\bullet\;} & \widetilde{R}^+_\bullet(M) \\
\Big\downarrow {\scriptstyle \not{\mathfrak{D}}_\bullet} & & \Big\downarrow {\scriptstyle \not{\mathfrak{D}}_\bullet} & & \Big\downarrow {\scriptstyle \not{\mathfrak{D}}_\bullet} \\
\Psi\mathrm{Dir}^\times_\bullet(M) & \xleftarrow[]{\;\simeq\;} & B_\bullet & \xrightarrow[\simeq]{\;\mathrm{susp}_\bullet\;} & \widetilde{\Psi\mathrm{Dir}}^\times_\bullet(M)
\end{array}
$$

is commutative up to homotopy.

Sketch of Proof

Comparison to KK-theory:

$$\begin{array}{ccc} \pi_n(B_\bullet) & \xrightarrow{\cong} & KK(\mathcal{C}_0(\mathbb{R}^{n+1}, Cl_{0,d}); \mathbb{R}) \\ \downarrow {\scriptstyle \pi_n(\mathrm{susp}_\bullet)} & & \cong \downarrow {\scriptstyle (-)\sharp[\not{D}]} \\ \pi_n(\widetilde{\Psi \mathrm{Dir}}^\times_\bullet(M^d)) & \longrightarrow & KK(Cl_{0,d+n+1}; \mathbb{R}) \end{array}$$

1. Construct horizontal maps
2. Upper horizontal is isomomorphism (Ebert/Kasparov)
3. Show commutativity
4. Compute $\pi_n(\widetilde{\Psi \mathrm{Dir}}^\times_\bullet(M^d))$ "by hand".

$\widetilde{R}^+_\bullet(M)$ is Kan $\qquad\qquad \widetilde{\Psi\mathrm{Dir}}^\times_\bullet(M) = KO^{-(d+1)}$

Thank you!

$$\begin{array}{ccc} R^+(M) & \overset{}{\hookrightarrow} & \widetilde{R}^+(M) \\ \Big\downarrow \scriptstyle \mathrm{inddif}_H & & \Big\downarrow \scriptstyle \mathrm{inddif}_{GL} \\ KO^{-(d+1)} & \overset{\simeq}{\dashrightarrow} & KO^{-(d+1)} \end{array}$$

$$\begin{array}{ccccc} R^+_\bullet(M) & \overset{\simeq}{\hookleftarrow} & A_\bullet & \xrightarrow{\mathrm{susp}_\bullet} & \widetilde{R}^+_\bullet(M) \\ \Big\downarrow \scriptstyle \not{\mathfrak{D}}_\bullet & & \Big\downarrow \scriptstyle \not{\mathfrak{D}}_\bullet & & \Big\downarrow \scriptstyle \not{\mathfrak{D}}_\bullet \\ \Psi\mathrm{Dir}^\times_\bullet(M) & \overset{\simeq}{\hookleftarrow} & B_\bullet & \underset{\simeq}{\xrightarrow{\mathrm{susp}_\bullet}} & \widetilde{\Psi\mathrm{Dir}}^\times_\bullet(M) \end{array}$$

Proof Sketch Factorisation Theorem

$$\mathrm{scal}(\mathrm{susp}(g)) = \mathrm{scal}(g) + \sum_{j=1}^{n} \frac{3}{4}|\partial_j g|^2 - \mathrm{tr}(\partial_j^2 g) - \frac{1}{4}\mathrm{tr}(\partial_j g)^2$$

$$\mathrm{susp}(P)^2 = P^2 \otimes \mathrm{id} - \sum_{j=1}^{n} \partial_j \cdot [P \otimes \mathrm{id}, \nabla_j^{\mathfrak{S}}] - \Delta_{\mathbb{R}^n} + \sum_{1\le j<k\le n} \mathfrak{R}(\partial_j, \partial_k)$$

$$\begin{aligned}\mathfrak{Err}(g) &:= \not{\mathfrak{D}}_{\mathrm{susp}(g)} - \mathrm{susp}(\not{\mathfrak{D}}_g) \\ &= \sum_{j=1}^{n} \mathrm{tr}(\partial_j g)\partial_j \cdot\end{aligned}$$

Block Dirac Operators

Example

The Dirac operator on $(M_1 \times M_2, g_1 \oplus g_2)$ decomposes:
$\not{\mathfrak{D}}_{g_1 \oplus g_2} = \not{\mathfrak{D}}_{g_1} \hat{\boxtimes} \mathrm{id} + \mathrm{id} \hat{\boxtimes} \not{\mathfrak{D}}_{g_2}$

Definition

Let $g \in \widetilde{R}_n(M)$ be a block metric on $M \times \mathbb{R}^n$. A symmetric pseudo Dirac operator P is a block Dirac operator if there are $r < R$ such that

(1) If $\mathrm{supp}(\sigma) \subset M \times R(-1,1)^n$, then $\mathrm{supp}(P\sigma) \subseteq M \times R(-1,1)^n$.

(2) $P - \not{\mathfrak{D}}_g$ is a bounded operator on $L^2(\mathfrak{S}_g)$.

(3) $P|_{\{\varepsilon x_i > r\}} = P_{(i,\varepsilon)} \hat{\boxtimes} \mathrm{id} + \mathrm{id} \hat{\boxtimes} \not{D}$.

All (invertible) block Dirac operators form a cubical set $\widetilde{\Psi\mathrm{Dir}}_\bullet^{(\times)}(M)$.

The Operator Suspension

Definition

For $P \in \Psi\mathrm{Dir}_n^{\times}(M)$, i.e.,

$P \colon \mathbb{R}^n \to \Psi\mathrm{Dir}^{\times}(M)$ smooth block map,

$g \colon \mathbb{R}^n \to \mathrm{Riem}(M)$ smooth block map of underlying metrics,

define on $\mathfrak{S}_{\mathrm{susp(g)}} - M \times \mathbb{R}^n$ the operator

$$\mathrm{susp}_n(P)(\sigma)(x,t) := P_t(\sigma(\,\cdot\,,t))(x) + \sum_{j=1}^{n} \partial_j \cdot (\nabla_{\partial_j}^{\mathfrak{S}_{\mathrm{susp}(g)}} \sigma)(x,t).$$

These maps assemble to a cubical map

$$\mathrm{susp}_\bullet \colon \Psi\mathrm{Dir}_\bullet^{\times}(M) \to \widetilde{\Psi\mathrm{Dir}}_\bullet(M).$$