Bismut-Zhang theorem for singular spaces

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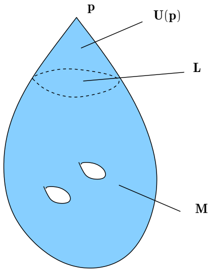
Singular spaces with isolated conical singularities (X^n, g^{TX})

For simplicity we assume that X has one single singularity p
$g^{TX} =$ smooth Riemannian metric on $X \setminus \{p\}$



$X = M \cup U(p)$

$M =$ compact manifold with boundary $\partial M = L$

$L =$ compact link manifold

$U(p) \simeq cL = \left(\mathbb{R}_+ \times L\right)/_{(0,x)\sim(0,y)}$

$g^{TX} = dr^2 + r^2 g^{TL}$ on $U(p)$

Let (F, ∇^F, g^F) be a flat Hermitian vector bundle on $X \setminus \{p\}$.

L^2-cohomology and spectral analysis

- (g^{TX}, g^F) induce an L^2-metric on k-forms with values in F
- complex of L^2-forms $(\mathcal{C}^\bullet, d_{\max})$ on $X \setminus \{p\}$, *i.e.*

$$\mathcal{C}^k := \{\alpha \text{ is a k-form with } \alpha \in L^2, d\alpha \in L^2\}$$

- L^2-cohomology: $H^\bullet_{(2)}(X, F) := H^\bullet(\mathcal{C}^\bullet, d_{\max})$
- the associated Laplacian is discrete

$$\Delta = (d_{\max} + \delta_{\min})^2 = \delta_{\min} d_{\max} + d_{\max}\delta_{\min}$$

- **Cheeger-Goresky-MacPherson ('80):**

$$H^\bullet_{(2)}(X, F) \simeq IH^\bullet_{\underline{m}}(X, F) \simeq \ker(\Delta)$$

Ray-Singer metric (analytic torsion)

... is a metric on

$$\det IH^\bullet_{\underline{m}}(X,F) := \det IH^0_{\underline{m}} \otimes (\det IH^1_{\underline{m}})^{(-1)} \otimes \ldots \otimes (\det IH^n_{\underline{m}})^{(-1)^{n-1}}$$

- L^2-metric on forms induces a metric on $\ker(\Delta)$, hence:

$$| \quad |^{RS}_{\det IH^\bullet_{\underline{m}}(X,F)}$$

- torsion zeta function:

$$\zeta(s) := \sum_{k=0}^{n} (-1)^{k+1} k \sum_{\lambda \in \mathrm{spec}(\Delta^{(k)}) \setminus \{0\}} \frac{1}{\lambda^s}, \ \mathrm{Re}(s) >> 0$$

Theorem (Dar '87)

The Ray-Singer metric (= analytic torsion) is welldefined:

$$\| \quad \|^{RS}_{\det IH^\bullet_{\underline{m}}(X,F)} := | \quad |^{RS}_{\det IH^\bullet_{\underline{m}}(X,F)} \exp\left(\frac{1}{2}\zeta'(0)\right).$$

Question

Does the Ray-Singer metric admit a topological interpretation?

$$\| \quad \|^{RS}_{\det IH^{\bullet}_{\underline{m}}(X,F)} = ?$$

.... for a smooth compact manifold

Cheeger-Müller Theorem ('78/'79)

Let (M^n, g^{TM}) a smooth compact Riemannian manifold, (F, ∇^F, g^F) a flat unitary vector bundle. Then

$$\| \quad \|^{RS}_{\det H^\bullet(M,F)} \text{ is a topological invariant.}$$

It is equal to the Reidemeister (or Milnor) metric.

- Müller '93: the same holds if (F, ∇^F, g^F) is unimodular

Bismut and Zhang ('92, '94)

For an arbitrary flat Hermitian vector bundle: comparison formula between Ray-Singer and Milnor metric

... on singular spaces with conical singularities

A.Dar: Does the Ray-Singer metric admit a topological interpretation?

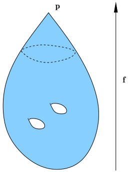
$$\| \quad \|^{RS}_{\det IH^{\bullet}_{\underline{m}}(X,F)} = ?$$

Three strategies of approach to Cheeger-Müller:

- **Glueing:** Vertman ('09, '12), Hartmann-Spreafico ('09-'17, '20 preprint), Vertman-Müller ('14), Lesch ('98, '13)
- **Degeneration:** for edge spaces, even codim. singularity: Albin-Rochon-Sher (to appear), Mazzeo-Vertman ('12)
- **Bismut-Zhang strategy:** L.

Witten deformation (ideas)

deformation of the L^2-complex $(\mathcal{C}^\bullet, d_{max})$ using a radial Morse function f:

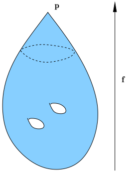


$$f_{sm} := f_{|X\setminus\{p\}} \quad = \quad \text{smooth Morse function}$$

$$f_{|cL} \quad = f(p) - \tfrac{1}{2}r^2$$

- deformation parameter T, later: $T \to \infty$
- the deformed L^2-complex $(\mathcal{C}_T^\bullet, d_{T,\max})$,
$$d_T = e^{-Tf} d e^{Tf} = d + Tdf$$
- the Witten Laplacian
$$\Delta_T = (d_{T,\max} + \delta_{T,\min})^2 = \Delta + T(\mathrm{Lie}_{\nabla f} + \mathrm{Lie}^*_{\nabla f}) + T^2|\nabla f|^2$$

Local model operator



$$f_{|X\setminus\{p\}} = \text{smooth Morse function}$$

$$f_{|cL} = f(p) - \tfrac{1}{2}r^2$$

Local model operator: $\Delta_T^{(k)} = \Delta^{(k)} + T(n-2k) + T^2 r^2$

Local Hodge-de Rham Thm: $\ker(\Delta_T^\bullet) \simeq IH^\bullet_{\underline{m}}(cL, L, F)$
forms in $\ker(\Delta_T^\bullet)$ have exponential decay

Torsion zeta function: ζ_{Δ_1} is holomorphic at 0.

Analytic torsion of the Witten Laplacian:

$$\| \quad \|^{RS}_{\det IH^\bullet_{\underline{m}}(cL,L,F)} := | \quad |^{RS}_{\det IH^\bullet_{\underline{m}}(cL,L,F)} \exp\left(\frac{1}{2}\zeta'_{\Delta_1}(0)\right)$$

Witten deformation (results)

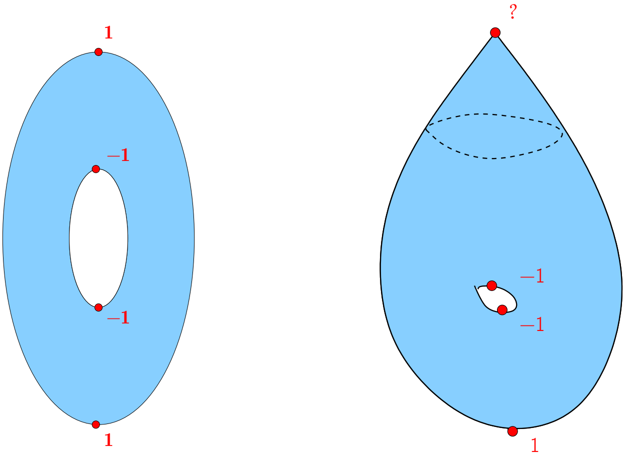
- Spectral Gap Theorem for Δ_T and Morse inequalities for $IH^\bullet(X,F)$ (**L. '17, Alvarez-Lopez and Calaza '17**)
- **L. '17**: Assuming the Morse-Smale condition, the Witten complex converges to a "singular" Morse-Thom-Smale complex $(IC^\bullet_{\underline{m}}(X,F,\nabla f),\partial^\bullet)$
- Using this complex and the analytic torsion of the model Witten Laplacian at p we define the

Bismut-Zhang metric $\|\quad\|^{BZ}_{\det IH^\bullet_{\overline{m}}(X,F)}$

smooth: Witten '82, Helffer/Sjöstrand '84-'85, Bismut/Zhang '92

most of the results above also hold for iterated conical stratified spaces

Summary (so far)



Cheeger-Müller Theorem for spaces with isolated conical singularities

Theorem (L. '20 Duke)

Let (X^n, g^{TX}) be a space with isolated conical singularties, (F, ∇^F, g^F) be a flat unitary vector bundle on $X \setminus \{p\}$, $f : X \longrightarrow \mathbb{R}$ a radial Morse function, such that $\nabla_g f$ is a standard Morse-Smale vector field. We assume in addition:

- (X^n, g^{TX}) is Witt (*i.e.* $H^{\frac{n-1}{2}}(L, F_L) = 0$) and a spectral Witt condition holds. In case n even, we assume that X is oriented.

Then:

$$\|\ \ \|^{RS}_{\det IH^\bullet_{\underline{m}}(X,F)} = \|\ \ \|^{BZ}_{\det IH^\bullet_{\underline{m}}(X,F)}.$$

Bismut-Zhang theorem for spaces with isolated conical singularities

Theorem (L. '22)

Let (X^n, g^{TX}) be a space with isolated conical singulararties, (F, ∇^F, g^F) be a arbitrary flat Hermitian vector bundle on $X \setminus \{p\}$, $f : X \to \mathbb{R}$ a radial Morse function, such that $\nabla_g f$ is a standard Morse-Smale vector field. Then:

$$\log\left(\frac{\|\quad\|^{RS}_{\det IH^{\bullet}_{\underline{m}}(X,F)}}{\|\quad\|^{BZ}_{\det IH^{\bullet}_{\underline{m}}(X,F)}}\right)^2 = -\int_X \theta(F, g^F)(\nabla_g f)^* \Psi(X, TX),$$

where

- $\theta(F, g^F)$= real closed 1-form measuring the obstruction to the existence of a flat volume form on F
- $\Psi(X, TX)$= Mathai-Quillen current on TX

Anomaly Formulas

We also give anomaly formulas for all three terms in the Bismut-Zhang theorem, for example:

Anomaly Formula for $\| \quad \|^{RS}_{\det IH^{\bullet}_{\bar{m}}(X,F)}$ (L.'22)

For $\mathbb{R} \ni l \to (g_l^{TX}, g_l^F)$ a family of metrics satisfying a spectral Witt condition:

$$\partial_l \log\left(\left(\| \quad \|^{RS}_{\det IH^{\bullet}_{\bar{m}}(X,F),l}\right)^2\right)$$
$$= \int_X \mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX, \nabla_l^{TX})$$
$$+ \int_X \frac{\partial}{\partial_l}\widetilde{e}(TX, \nabla_0^{TX}, \nabla_l^{TX})\theta(F, g_l^F)$$
$$+ \text{ local term of p}$$