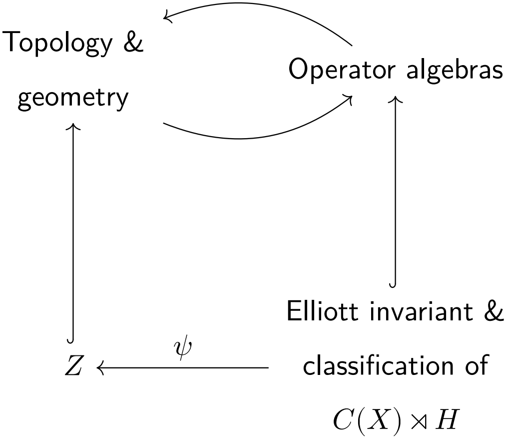
Classification via index theory on the mapping torus

Valerio Proietti (Hao Guo, Hang Wang)

https://vproietti.gitlab.io

East China Normal University



What is Z ? What is ψ ?

What is Z ? (Elliott invariant)

The **quotient**, the **leaf**, the **operator**

- K-theory of the homotopy quotient Y associated to the H-action on X
- map induced in K-theory by the inclusion of the fundamental domain inside Y
- index of Dirac operator twisted by bundles over Y

What is Z ? (Classification of $C(X) \rtimes H$)

Lifting "equivariant" homotopy equivalences of homotopy quotients to isomorphisms of associated "noncommutative spaces"

- equivariant: structure preserving, e.g., foliation, orbifold
- noncommutative space: crossed product C^*-algebra
- lifting homotopy to actual homeo/diffeomorphism?
  (probably hopeless in general)

What is ψ ?

The standard tools of **noncommutative geometry**

- Green-Rieffel Morita equivalence (Induction)
- Connes' Thom isomorphism (Baum–Connes conjecture)
- index theorem on foliated spaces

X compact Hausdorff space, **finite** covering dimension

G simply connected solvable Lie group (e.g., Heisenberg group)

$H \subseteq G$ discrete and co-compact subgroup

$H \to \mathrm{Homeo}(X)$ **free** and **minimal** action

free: stabilizers are trivial

minimal: H-orbits are dense

Simplify: $(H, G) = (\mathbb{Z}^d, \mathbb{R}^d)$

$\mathcal{H} = H \ltimes X$ principal étale groupoid

$\mathcal{H}$ is amenable (solvable groups are amenable)

$A = C^*(\mathcal{H}) = C(X) \rtimes H$ simple (minimality), nuclear and UCT-class (amenability)

$\mathcal{H}$ is free

$$\Downarrow$$

invariant Borel probability measure on $X \leftrightarrow$ tracial state on A

$$\tau_\mu(f) = \int_X f|_X \, d\mu$$

Define through diagonal action $Y = \dfrac{X \times EH}{H}$ (compact)

EH is contractible, Y is **homotopy quotient**

G is diffeomorphic to $\mathbb{R}^d$

$H \ltimes G$ is free and proper $\implies$ $EH = G$ and $Y = \mathrm{Ind}_H^G(X)$

Simplified setting: $Y = \dfrac{X \times \mathbb{R}^d}{\mathbb{Z}^d}$ is called **mapping torus**

$X \times (0,1)^d \hookrightarrow Y$ is **fundamental domain**

The **quotient**, the **leaf**, the **operator**

Y is a **foliated space**

local structure: $X \times \mathbb{R}^d =$ transversal $\times$ leaf

Each leaf is dense in Y (minimality)

$\mathcal{R}(Y) = \{(x, y) \in Y \times Y \mid x$ and y lie on same leaf$\}$

$Y = \mathrm{Ind}_H^G(X) = X \times_H G$ carries G-action (right translation),

$$\mathcal{G} = G \ltimes Y$$

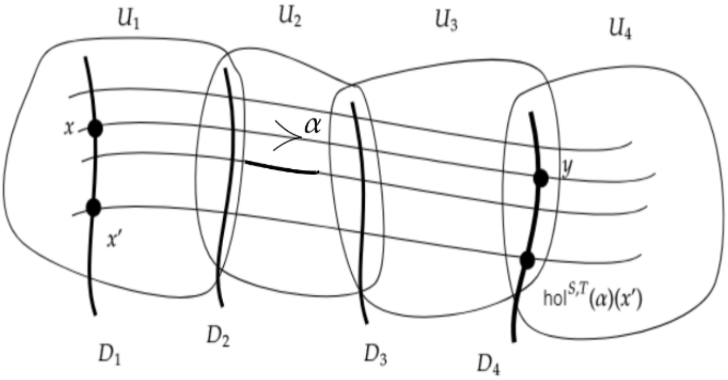
$\mathcal{H}$ is free $\implies \mathcal{R}(Y) \cong \mathcal{G}$

The **quotient**, the **leaf**, the **operator**

Connes' work: the **holonomy groupoid** associated to the foliation $(Y, \mathcal{R}(Y))$ is the "noncommutative space" of the foliation

$\implies K_0(C_r^*(\mathrm{Hol}(Y))$ is the receptacle for index classes of elliptic differential tangential operators on Y

tangential: family of operators each acting on leaf, parameterized by transversal, and $\mathrm{Hol}(Y)$-invariant



Leaf contractible $\implies \mathrm{Hol}(Y) \cong \mathcal{R}(Y) \cong \mathcal{G}$

Recall $\mathcal{H} = H \ltimes X$ and $\mathcal{G} = G \ltimes Y$

We want to **classify** $A = C(X) \rtimes H = C^*(\mathcal{H})$

Connection to foliation C^*-algebra $C^*(\mathrm{Hol}(Y)) \cong C^*(\mathcal{G})$?

Idea (geometry): first return map, restriction to transversal (e.g.,

Kronecker foliation and irrational rotation algebra)

define **principal bibundle** from transversal

$\mathrm{Res}_Y^X(\mathcal{G}) \cong \mathcal{H} \implies \mathcal{H}$ is Morita equivalent to $\mathcal{G}$

bijection: $\mathrm{Traces}(C^*(\mathcal{H})) \leftrightarrow \mathrm{Traces}(C^*(\mathcal{G}))$

"invariant transverse measures"

Recall $Y = X \times_H G = \mathrm{Ind}_H^G(X)$

Green-Rieffel machine: $C^*(G \ltimes \mathrm{Ind}_H^G(X)) \otimes \mathbb{K} \cong C^*(H \ltimes X) \otimes \mathbb{K}$

We want to **classify** $A = C(X) \rtimes H = C^*(\mathcal{H})$

Recall $\mathcal{G} = G \ltimes Y$ and G is solvable simply connected Lie group

C^*(group extension) $\cong$ twisted crossed product $C^*(N_1) \rtimes_\sigma N_2$

$N_2 = \mathbb{R}$ can untwist $\implies$ iterated crossed product by $\mathbb{R}$

Bottom line: can use Connes' Thom isomorphism

(more general: Baum-Connes conjecture?)

$K_*(A) \cong_{\mathsf{Mor}} K_*(C^*(\mathcal{G})) \cong_{\mathsf{TC}} K^{*+d}(Y)$

We want to **classify** $A = C(X) \rtimes H = C^*(\mathcal{H})$

X	Y
A	$C^*(\mathcal{G})$
$\mathrm{Traces}(A)$	inv. trans. measures
$K_*(A)$	$K^{*+d}(Y)$
$[1]_0$	K^*(fund. domain) $\to K^*(Y)$
trace pairing	index pairing

Suppose you have $A_\alpha = C(X) \rtimes_\alpha H$ and $A_\beta = C(X) \rtimes_\beta H$

$F : Y_\alpha \to Y_\beta$ homotopy equivalence

inv. trans. measures $= M(Y)$

$$\implies F_* \colon M(Y_\alpha) \cong M(Y_\beta)$$

idea: cohomology + **Riesz representation** theorem

$$C \colon M(Y) \cong H^d_\tau(Y, \mathbb{R})^*$$

d-form $\omega \leftrightarrow$ tangentially smooth 1-density $\leftrightarrow$ tangential measure

tan. measure: section $\ell \mapsto \lambda^{(\ell)}_\omega$ from leaf space to Radon measures

$$\tilde{\mu}(E \subseteq Y) = \int_E \lambda \, d\mu := \int \int_{f^{-1}(x)} \chi_E(y) \, d\rho_x(y) d\mu(x)$$

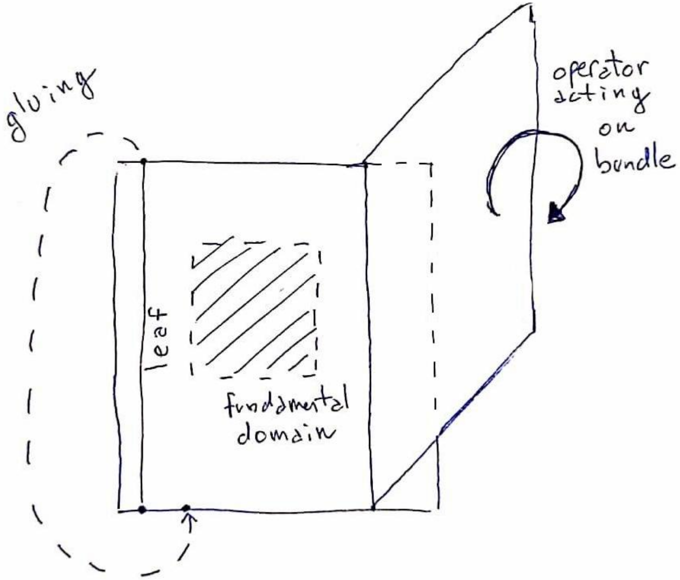
$$
\begin{array}{ccccccc}
\mathbb{R} & \xleftarrow{\tau_\eta} & K_0(C^*(\mathcal{H}_\alpha)) & \xleftarrow{\Phi_\alpha^{-1}F^*\Phi_\beta} & K_0(C^*(\mathcal{H}_\beta)) & \xrightarrow{\tau_\mu} & \mathbb{R} \\
\Vert & & \downarrow{\scriptstyle \mathsf{Mor}_\alpha} & & \downarrow{\scriptstyle \mathsf{Mor}_\beta} & & \Vert \\
\mathbb{R} & \xleftarrow{\tilde{\tau}_\eta} & K_0(C^*(\mathcal{G}_\alpha)) & & K_0(C^*(\mathcal{G}_\beta)) & \xrightarrow{\tilde{\tau}_\mu} & \mathbb{R} \\
\Vert & & \downarrow{\scriptstyle TC_\alpha} & & \downarrow{\scriptstyle TC_\beta} & & \Vert \\
\Vert & & K^d(Y_\alpha) & \xleftarrow{F^*} & K^d(Y_\beta) & & \Vert \\
\Vert & & \downarrow{\scriptstyle \mathrm{Ch}} & & \downarrow{\scriptstyle \mathrm{Ch}} & & \Vert \\
\mathbb{R} & \xleftarrow{C(\eta)} & H_\tau^{[d]}(Y_\alpha,\mathbb{R}) & \xleftarrow{F^*} & H_\tau^{[d]}(Y_\beta,\mathbb{R}) & \xrightarrow{C(F_*\eta)} & \mathbb{R}
\end{array}
$$

sides commute: index theorem

Conclusion: trace pairing $\leftrightarrow$ index pairing

The **quotient**, the **leaf**, the **operator**

$K_*(A)$	$K^{*+d}(Y)$	**quotient**
$[1]_0$	$K^*(\text{fund. domain}) \to K^*(Y)$	**leaf**
$\mathrm{Traces}(A)$	inv. trans. measures	**leaf**
trace pairing	index pairing	**operator**


gluing
operator acting on bundle
leaf
fundamental domain

Recall $A_\alpha = C(X) \rtimes_\alpha H$ and $A_\beta = C(X) \rtimes_\beta H$

inclusion of fundamental domain: $\iota\colon Y^0 \to Y$

Theorem

$A_\alpha \cong A_\beta$ *if and only if* $\mathrm{GEll}(A_\alpha) \cong \mathrm{GEll}(A_\beta)$

$\mathrm{GEll}(A) = (K^*(Y), K^d(\iota), H_d^\tau(Y, \mathbb{R}),$ *index pairing*$)$

Corollary

$F\colon Y_\alpha \to Y_\beta$ *"nice" homotopy equivalence can be* ***lifted*** *to*

isomorphism *of "noncommutative spaces"*

$$C^*(\mathcal{H}_\alpha) \cong C^*(\mathcal{H}_\beta)$$

$$C^*(\mathcal{G}_\alpha) \otimes \mathbb{K} \cong C^*(\mathcal{G}_\beta) \otimes \mathbb{K}$$

Contributions

- Comparison homotopy quotient vs crossed product
- Geometric perspective on Elliott invariant and classification
- Classification via ordered K-theory
- Characterization of $\mathbb{Z}^d$-odometers with application to quasi-crystals