Heat asymptotics and Weyl law for locally symmetric manifolds of finite volume

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Connections with

- Representation theory of reductive groups over local and global fields.
- Partial differential equations on locally symmetric spaces.
- Number theory.

I. Heat expansion and Weyl law

- X compact Riemannian manifold without boundary, $\dim X = n$.
- $E \to X$ Hermitean vector bundle, ∇_E Hermitean connection in E.
- $\Delta_E := \nabla_E^* \nabla_E$ Bochner-Laplace operator, acting in $C^\infty(X, E)$.
- $0 \le \lambda_1 < \lambda_2 < \cdots \to \infty$ eigenvalues of Δ_E, $m(\lambda_i)$ multiplicity of λ_i.

Heat expansion

$$\mathrm{Tr}(e^{-t\Delta_E}) = \sum_{j=1}^{\infty} m(\lambda_j) e^{-t\lambda_j} \sim t^{-n/2} \sum_{k \ge 0} a_k t^k, \quad t \to 0,$$

$$a_0 = \frac{\mathrm{rank}(E)\,\mathrm{vol}(X)}{(4\pi)^{n/2}}.$$

- $N_X(\lambda; E) := \sum_{\lambda_j \leq \lambda} m(\lambda_j)$ eigenvalue counting function.

Weyl law

Karamata's theorem gives

$$N_X(\lambda; E) = \frac{\operatorname{rank}(E)\operatorname{vol}(X)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}\lambda^{n/2} + o(\lambda^{n/2}), \quad \lambda \to \infty.$$

Wave equation

- Estimation of the remainder term.

Avakumovic, Hörmander E trivial,

$$N_X(\lambda) = \frac{\operatorname{vol}(X)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}\lambda^{n/2} + O(\lambda^{(n-1)/2}), \quad \lambda \to \infty.$$

- Remainder term optimal.
- X non-compact: No general results can be expected.
- Results exist for manifolds with special end structure.

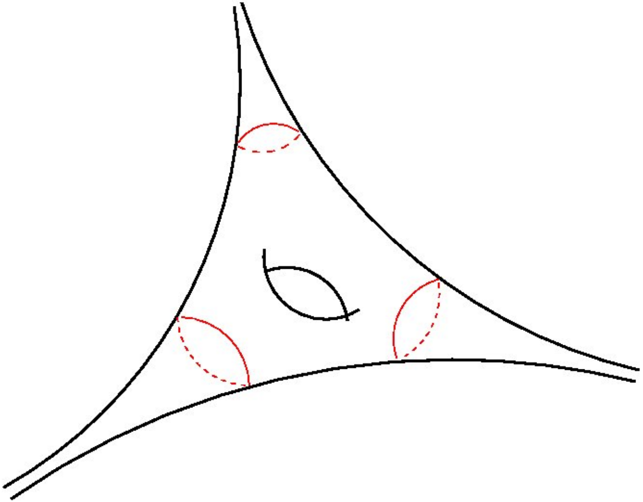
II. Locally symmetric spaces

- G semisimple real Lie group, non-compact type, finite center.
- $K \subset G$ maximal compact subgroup.
- $\widetilde{X} = G/K$ Riemannian symmetric space of non-positive curvature, equipped with G-invariant metric.
- $\Gamma \subset G$ lattice, i.e., a discrete subgroup of G with $\mathrm{vol}(\Gamma \backslash G) < \infty$.
- Γ acts properly discontinously on $\widetilde{X}$.
- $X = \Gamma \backslash \widetilde{X} = \Gamma \backslash G/K$ locally symmetric space.

Example: $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) \cong \mathbb{H}^2 = \{z \in \mathbb{C} \colon \ \mathrm{Im}(z) > 0\}$.
For $N \in \mathbb{N}$ let

$$\Gamma(N) = \left\{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \colon \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\right\}.$$

- $\Gamma(N)\backslash\mathbb{H}^2$ Riemann surface, non-compact, finite area.



Surface of genus 1 with 3 cusps

III. Locally homogeneous vector bundles

- $\sigma\colon K \to V_\sigma$ unitary representation of K, $\dim V_\sigma < \infty$
- Right K-action on $G \times V_\sigma$ by $(g, v) \cdot k = (gk, \sigma(k)^{-1} v)$.
- $\widetilde{E}_\sigma := (G \times V_\sigma)/K$ homogeneous vector bundle with canonical connection ∇^σ.
- $E_\sigma := \Gamma \backslash \widetilde{E}_\sigma$ locally homogeneous vector bundle.
- Bochner-Laplace operator

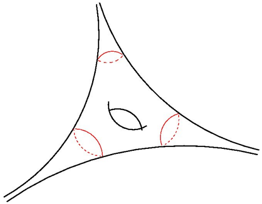
$$\Delta_\sigma = (\nabla^\sigma)^* \nabla^\sigma \colon C^\infty(X, E_\sigma) \to C^\infty(X, E_\sigma)$$

- X non-compact, $\mathrm{vol}(X) < \infty$. Spectrum of Δ_σ consists of a pure point spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and a continuous spectrum described by Eisenstein series.

Problem: $N_{\mathrm{pp}}(\lambda, \sigma)$ eigenvalue counting function. Estimation of the growth of $N_{\mathrm{pp}}(\lambda, \sigma)$ as $\lambda \to \infty$. There is a refined version of the problem.

IV. Hyperbolic surfaces

- $\mathbb{H}^2$ upper half plane, $\mathbb{H}^2 \cong \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$.
- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ discrete, torsion free subgroup with $\mathrm{vol}(\Gamma \backslash \mathrm{SL}(2,\mathbb{R}) < \infty$.
- $X := \Gamma \backslash \mathbb{H}^2$ hyperbolic surface of finite area.



- $X = X_0 \cup Y_1 \cup \cdots \cup Y_m$, X_0 compact surface with boundary
- $Y_i \cong [a_i, \infty) \times S^1$ cusp, metric on Y_i has the form

$$h|_{Y_i} = \frac{dy_i^2 + dx_i^2}{y_i^2}, \quad (y_i, x_i) \in Y_i.$$

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad z = x + iy.$$

Hyperbolic Laplace operator.

- $\Delta \colon C^\infty(X) \to C^\infty(X)$ is essentially self-adjoint in $L^2(X)$.

Theorem (Selberg, Roelcke, 1954):

1) $\mathrm{Spec}_{pp}(\Delta)\colon 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, eigenvalues of finite multiplicities,
2) If $\Gamma\backslash\mathbb{H}^2$ is non-compact, $\mathrm{Spec}_{ac}(\Delta) = [1/4, \infty)$,
3) $L^2_{ac}(\Gamma\backslash\mathbb{H}^2)$ can be described in terms of Eisenstein series.

Eisenstein series Example $\Gamma = \mathrm{SL}(2,\mathbb{Z})$:

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma(z))^s = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}} \frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

- $E(z,s)$ admits meromorphic extension to $\mathbb{C}$, holomorphic for $\mathrm{Re}(s) = 1/2$.
- $\Delta E(z,s) = s(1-s)E(z,s)$
- $r \in \mathbb{R} \mapsto E(z, 1/2 + ir)$ generalized eigenfunction.
- $\int_0^1 E(x+iy,s)dx = y^s + c(s)y^{1-s}$
- $c(s)$ "scattering matrix".

Cuspidal eigenfunctions. $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

- $f \in L^2(X)$, $\Delta f = \lambda f$, assume that f is symmetric w.r.t. the reflection $x+iy \mapsto -x+iy$.

Since f satisfies $f(z+1) = f(z)$ it admits the following Fourier expansion w.r.t. x: Let $\lambda = s(1-s)$, $\mathrm{Re}(s) \geq 1/2$.

$$f(x+iy) = a_0 y^s + b_0 y^{1-s} + \sum_{n=1}^{\infty} a_n y^{1/2} K_{s-1/2}(2\pi n y)\cos(2\pi n x),$$

- If $\int_0^1 f(x+iy)dx = 0$, then f is called cusp function.
- Fourier expansion implies that for a cuspidal eigenfunction there exists $c > 0$ such that

$$f(x+iy) \ll e^{-cy}, \quad y \to \infty.$$

- If $\lambda \geq 1/4$, then $s = 1/2 + ir$, $r \in \mathbb{R}$, $y^{1/2+ir} \notin L^2$, $\Rightarrow$ $a_0 = b_0 = 0$.
- $\Rightarrow$ every L^2-eigenfunction of Δ with eigenvalue $\lambda \geq 1/4$ is a cusp function.
- $L^2_{\mathrm{cus}}(X) \subset L^2_{pp}(X)$ subspace spanned by cusp functions.

Residual eigenfunctions

- $X = \Gamma \backslash \mathbb{H}^2$ general hyperbolic surface, $E_k(z, s)$, $k = 1, ..., m$, Eisenstein series attached to the k-th cusp of X.
- Poles of $E_k(z, s)$ in $\mathrm{Re}(s) \geq 1/2$ are simple and are contained in $(1/2, 1]$

- $\sigma \in (1/2, 1]$ pole of $E_k(z,s)$. Let $\phi := Res_{s=\sigma} E(z,s)$. Then $\phi \in L^2(X)$ and $\Delta\phi = \sigma(1-\sigma)\phi$.
- $L^2_{\mathrm{res}}(X) \subset L^2_{pp}(X)$ subspace spanned by residues of Eisenstein series. Then

$$L^2_{pp}(X) = L^2_{\mathrm{cus}}(X) \oplus L^2_{\mathrm{res}}(X), \quad \dim L^2_{\mathrm{res}}(X) < \infty.$$

The cut-off Laplacian Lax-Phillips, Colin de Verdiere.
Assume that X has a single cusp. $X = X_0 \cup Y$, $Y = [1,\infty) \times S^1$. For $a > 1$ let

$$H^1_a(X) := \{f \in H^1(X) \colon f_0|_{(a,\infty)} = 0\},$$

wherre f_0 is the 0-th Fourier coefficient defined in the distributional sense.

- $q_a \colon H^1_a(X) \to \mathbb{R}$, $q_a(f) = \| \operatorname{grad} f \|^2$.
- Δ_a self-adjoint operator associated to the quadratic form q_a.

Theorem (Colin de Verdière)

- Δ_a has pure point spectrum $0 \le \Lambda_1 \le \Lambda_2 \le \cdots$.
- Let $N_a(\lambda) = \#\{j \colon \Lambda_j \le \lambda\}$. Then

$$N_a(\lambda) \cong \frac{\mathrm{Area}(X)}{4\pi}\lambda, \quad \lambda \to \infty.$$

- $f \in L^2_{\mathrm{cus}}(X)$, $\Delta f = \lambda f$. Then $f \in \mathrm{dom}(\Delta_a)$ and $\Delta_a f = \lambda f$.

Corollary

$$\limsup_{\lambda\to\infty} \frac{N_X(\lambda)}{\lambda} \le \frac{\mathrm{Area}(X)}{4\pi}.$$

Theorem (Colin de Verdière) h_0 hyperbolic metric, $f \in C_c^\infty(X)$, $h_f := e^f h_0$ conformal deformation. For generic f, Δ_f has no embedded eigenvalues, i.e., no eigenvalues $\lambda \ge 1/4$.

Conjecture (Phillips-Sarnak). Except for the Teichmüller space of the once punctured torus, for a generic Γ, the Laplacian on $\Gamma\backslash\mathbb{H}^2$ has only a finite number of eigenvalues.

- For arithmetic groups like $\Gamma(N) \subset \mathrm{SL}(2,\mathbb{Z})$, eigenfunctions are of significance in number theory.

Example $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $f \in L^2(\Gamma\backslash\mathbb{H}^2)$, such that

- $\Delta f = (1/4 + r^2) f$.
- $T_n f = \lambda_n f$, T_n Hecke operator, $f(z) = f(-\bar{z})$.

Then

$$f(z) = \sum_{n=1}^{\infty} a_n y^{1/2} K_{ir}(2\pi n y) \cos(2\pi n x).$$

Then

$$L(s; f) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - a_p p^{-s} + p^{-2s}\right)^{-1}$$

is an automorphic L-function. Satisfies functional equation, etc.

Selberg trace formula

- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ a lattice,
- $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ, $\lambda_j = 1/4 + r_j^2$, $r_j \in \mathbb{C}$, $\arg(r_j) \in \{0, \pi/2\}$.
- $C(s)$ scattering matrix, $\phi(s) = \det C(s)$.
- $g \in C_c^\infty(\mathbb{R})$, $h(z) = \int_{\mathbb{R}} g(r) e^{-irz} dr$.

$$\sum_j h(r_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\phi'}{\phi}(1/2+ir)\,dr + \frac{1}{4}\phi(1/2)h(0)$$

$$= \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H}^2)}{4\pi}\int_{\mathbb{R}} h(r) r \tanh(\pi r)\,dr$$

$$+ \sum_{\{\gamma\}_\Gamma} \frac{l(\gamma_0)}{2\sinh\left(\frac{l(\gamma)}{2}\right)} g(l(\gamma))$$

$$-\frac{m}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr + \frac{m}{4}h(0) - m\ln 2\, g(0).$$

- We apply the trace formula to the heat operator $e^{-t\Delta}$.

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{\phi'}{\phi}(1/2+ir)dr$$
$$= \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H}^2)}{4\pi t} + \frac{a\log t}{\sqrt{t}} + \frac{b}{\sqrt{t}} + O(1)$$

as $t \to 1$ for some $a, b \in \mathbb{R}$. Let

$$N_\Gamma(\lambda) = \#\{j\colon \lambda_j \le \lambda^2\}, \quad M_\Gamma(\lambda) = -\frac{1}{4\pi}\int_{-\lambda}^{\lambda}\frac{\phi'}{\phi}(1/2+ir)dr.$$

Lemma $M_\Gamma(\lambda)$ is monotonic increasing for $\lambda \gg 0$.

$\Rightarrow$ Karamata's theorem implies

Theorem (Selberg)

$$N_\Gamma(\lambda) + M_\Gamma(\lambda) = \frac{\mathrm{Area}(\Gamma \backslash \mathbb{H}^2)}{4\pi}\lambda^2,$$

as $\lambda \to \infty$.

- A more sophisticated use of the trace formula gives an estimation of the remainder term of order $O(\lambda \log \lambda)$. This is essentially Hörmander's method.
- The winding number $M_\Gamma(\lambda)$ counts the number of resonances with their order, i.e., poles of $\det C(s)$ in a circle of radius λ. This makes it clear that $N_\Gamma(\lambda) + M_\Gamma(\lambda)$ is stable under perturbations.
- This remains true, if we move outside hyperbolic metrics and consider surfaces with hyperbolic cusps. Recent work of Bonthonneau: Two term asymptotic expansion for manifolds with cusps and negatice curvature.
- In general, $N_\Gamma(\lambda)$ and $M_\Gamma(\lambda)$ can not be separated.

Arithmetic groups

- $\Gamma = \mathrm{SL}(2, \mathbb{Z})$: Then the scattering matrix is given by

$$c(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

  where $\zeta(s)$ is the Riemann zeta function.
- For $\Gamma(N)$, coefficients of the scattering matrix $C(s)$ are given by fractions of Dirichlet L-functions $L(s,\chi)$.
- Using standard estimates of $L(s,\chi)$ and $\Gamma(s)$, it follows that the growth of the winding number of $\phi(s)$ is of lower order.

Theorem (Selberg) For every $n \in \mathbb{N}$ we have

$$N_{\Gamma(N)}(\lambda) = \frac{\mathrm{Area}(\Gamma(N)\backslash\mathbb{H}^2)}{4\pi}\lambda^2 + O(\lambda \log \lambda).$$

as $\lambda \to \infty$.

V. Higher rank

- $X = \Gamma\backslash G/K$, G semisimple, $K \subset G$ maximal compact, $\sigma \in \Pi(K)$, $E_\sigma \to X$ locally homogeneous vector bundle.
- $\Delta_\sigma = (\nabla^\sigma)^*\nabla^\sigma$ Bochner-Laplace operator.
- $C^\infty(X, E_\sigma) \cong (C^\infty(\Gamma\backslash G) \otimes V_\sigma)^K$. Similar for L^2
- $\phi \in L^2(\Gamma\backslash G)$ cusp form, if ϕ is $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$-finite, right K-finite and for every proper cuspidal parabolic subgroup $P \subset G$, N_P nilradical of P, one has

$$\int_{(\Gamma\cap N_P)\backslash N_P} \phi(ng)dn = 0.$$

- $L^2_{\rm cus}(X, E_\sigma) \subset L^2_{\rm pp}(X, E_\sigma)$ subspace, spanned by cusp forms. $L^2_{\rm res}(X, E_\sigma)$ orthogonal complement. "Residual subspace".
- $N_{\rm cus}(\lambda, \sigma)$ and $N_{\rm res}(\lambda, \sigma)$ corresponding eigenvalue counting functions.

General results

Theorem (Langlands, 1976) $L^2_{\mathrm{res}}(X, E_\sigma)$ is spanned by iterated residues of Eisenstein series.

Theorem (Donnelly, 1982) Let $n = \dim X$.

$$\limsup_{\lambda\to\infty} \frac{N_{\mathrm{cus}}(\lambda;\sigma)}{\lambda^{n/2}} \le \frac{\dim(\sigma)\,\mathrm{vol}(X)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}.$$

- Generalization of Colin de Verdière's method.

Theorem (Mü., 1989)

$$N_{\mathrm{res}}(\lambda,\sigma) \ll 1 + \lambda^{2n}, \quad \lambda \le 0.$$

- Method uses the Theorem of Langlands combined with a generalization of the cut-off Laplacian.

The Weyl law holds in the following cases:

i) $\sigma = 1$, no estimation of the remainder term.

- S. Miller: $X = \mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.
- E. Lindenstrauss, A. Venkatesh: $\mathbf{G}$ split adjoint semisimple group over $\mathbb{Q}$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ congruence subgroup, $G = \mathbf{G}(\mathbb{R})$, $X = \Gamma\backslash G/K$. **Method:** Use of Hecke operators.

ii) $\sigma \in \Pi(K)$ arbitrary, no estimation of the remainder term.

Theorem, (Matz, Mü., 2022): Let $\mathbf{G}$ be one of the following algebraic groups:

1. Inner form of $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$.
2. Symplectic, special orthogonal or unitary group.

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a congruence subgroup and $G = \mathbf{G}(\mathbb{R})$. Then the Weyl law holds for $\Gamma\backslash G/K$.

Methods:

- Use Arthur trace formula to show that
$$\sum_j e^{-t\lambda_j(\sigma)} = \operatorname{Tr}\left(e^{-t\Delta_{\sigma,\mathrm{pp}}}\right) \sim \frac{\dim(\sigma)\operatorname{vol}(X)}{(4\pi)^{n/2}} t^{-n/2}$$
as $t \to 0$.
- Main ingrdient of the proof: Study of automorphic L-functions that appear in the scattering matrices.

Example: $X = \mathrm{SL}(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$. $\partial(\overline{X}^{BS})$ has two components of maximal dimension Y_{P_1} and Y_{P_2}, which are torus fibrations

$$\pi_i \colon Y_{P_i} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2.$$

The associated scattering matrix $c_{P_2|P_1}(s)$, $s \in \mathbb{C}$, acts in the space of automorphic cusp forms on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$.

Let $\phi \in L^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2)$ be a non-constant even eigenfunction of Δ. Then

$$\phi(x+iy) = \sum_{n=1}^{\infty} a_n y^{1/2} K_{ir}(2\pi ny)\cos(2\pi nx),$$

Let

$$\Lambda(s,\phi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \mathrm{Re}(s) > 1,$$

be the L-function attached to ϕ.
Let $\lambda = 1/4 + r^2$, $r \in \mathbb{R}$. Let

$$\Lambda(s,\phi) = \pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-r}{2}\right)L(s,\phi).$$

be the completed L-function. Then

$$c_{P_2|P_1}(s)\phi = \frac{\Lambda(s,\phi)}{\Lambda(s+1,\phi)}\phi.$$

- $\Lambda(s,\phi)$ satisfies a functional equation and has similar properties as the Riemann zeta function. Can be used to estimate the contribution of the continuous spectrum to the trace formula.

iii) $\sigma = 1$, estimation of the remainder term.

Let $S_n = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n), \quad n \geq 2$, and let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a congruence subgroup. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ_Γ in $L^2(\Gamma \backslash S_n)$.

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}.$$

Thoerem 2 (Lapid, Mü.): Let $d = \dim S_n$, $N \geq 3$. Then

$$N_{\Gamma(N)}(\lambda) = \frac{\mathrm{Vol}(\Gamma(N)\backslash S_n)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^d + O(\lambda^{d-1}(\log\lambda)^{\max(n,3)})$$

as $\lambda \to \infty$.

Theorem 3 (Finis, Lapid, 2019): Let G be a simply connected, simple Chevalley group. Then there exists $\delta > 0$ such that for any congruence subgroup Γ of $G(\mathbb{Z})$ one has

$$N_{X,\mathrm{cus}}(\lambda) = \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\lambda^d + O_\Gamma(\lambda^{d-\delta}), \quad \lambda \geq 1,$$

where $X = \Gamma\backslash G(\mathbb{R})/K$, $d = \dim(X)$.