

Differential K-Theory and localization for Eta Invariant

Xiaonan Ma

Université Paris Cité

Institut de Mathématiques de Jussieu-Paris Rive Gauche

— joint work with Bo Liu (East China Normal University, China)

July 1 2022

Dirac operators in Topology, Geometry
and Representation Theory
Cortona, Italy

Contents

- 1 Equivariant eta Invariants and localization
- 2 Differential K-Theory and eta invariant

Dirac Operator on Spin^c Manifold

- (Y, g^{TY}) : oriented Riemannian manifold of dim n .
- Existence of spin^c structure on $Y \iff$ topological condition : second Stiefel-Whitney class of TY :
 $w_2(TY) = c_1(L) \in H^2(Y, \mathbb{Z}_2)$, with L complex line bundle on Y .
- Assume Y is spin^c , the spinor bundle over Y formally is

$$\mathcal{S} = \mathcal{S}(TY, L) = S \otimes L^{1/2},$$

and both classical spinor S and $L^{1/2}$ need not exist.

- Let (E, h^E) be a Hermitian complex vector bundle over Y , with Hermitian connection ∇^E .
- **Dirac operator** : $\{e_i\}$ any orthonormal frame of TY ,

$$D^Y \otimes E := \sum_i c(e_i) \nabla_{e_i}^{S \otimes E} : \mathcal{C}^\infty(Y, S \otimes E) \rightarrow \mathcal{C}^\infty(Y, S \otimes E).$$

$D^Y \otimes E$ is a 1st order, self-adjoint, elliptic differential operator.

Equivariant Eta Invariant

- Y : odd-dim. compact G -Spin^c manifold for compact Lie group G .
- $\underline{E} = (E, h^E, \nabla^E)$ G -equivariant on Y .
- Then $D^Y \otimes E$ commutes with the G -action.
Its eigenspace \mathcal{E}_{λ_i} with eigenvalue $\{\lambda_i\}$.
- $\eta_g(s) := \sum_{\lambda_i > 0} \frac{\text{Tr} |_{\mathcal{E}_{\lambda_i}} [g]}{|\lambda_i|^s} - \sum_{\lambda_j < 0} \frac{\text{Tr} |_{\mathcal{E}_{\lambda_j}} [g]}{|\lambda_j|^s}, \quad g \in G, s \in \mathbb{C}.$
- (Donnelly '78) $\eta_g(s)$ is holomorphic for $\text{Re}(s) \gg 0$ and holomorphic at 0 after meromorphic continuation.
- **Equivariant reduced APS eta invariant** :

$$\bar{\eta}_g(Y, \underline{E}) := \frac{1}{2} \eta_g(0) + \frac{1}{2} \text{Tr} |_{\ker(D^Y \otimes E)} [g] \in \mathbb{C},$$

is the boundary term of the equivariant APS index theorem with global (APS) boundary condition.

Localization of Eta Invariant ?

- How to understand $\bar{\eta}_g$ as a function on G and its relation on the contribution on the fixed point set Y^g (i.e., localization formula) ?
- Example : $Y = S^1$, \underline{E} trivial line bundle, $G = S^1$,
 $g.e^{i\theta} := e^{2\pi ikt+i\theta}$, $g = e^{2\pi it} \in S^1$, $k \in \mathbb{N}^*$.
- $Y^{S^1} := \{y \in Y : gy = y, \forall g \in S^1\} = \emptyset$, $A = \{g : g^k = 1\}$. Then

$$\bar{\eta}_g(Y) = \begin{cases} \frac{1}{1-g^k} & \text{if } g \in S^1 \setminus A; \\ \frac{1}{2} & \text{if } g \in A. \end{cases}$$

$\bar{\eta}_g$ is not even continuous on $G = S^1$!

- Theorem (Liu-Ma 2020) Y odd-dim. compact S^1 -Spin^c manifold. If $Y^{S^1} = \emptyset$, then $\bar{\eta}_g(\underline{TY}, \underline{E})$ as a function on $S^1 \setminus A$ with finite set $A = \{g \in S^1 : Y^g \neq \emptyset\}$, is the restriction of a rational function on S^1 with integral coefficients and without poles on $S^1 \setminus A$.

Localization of Eta Invariant II

- If $Y^{S^1} \neq \emptyset$, localization of eta invariants? Much hard to explain our precise result.
- Some localization formulas :
Berline-Vergne localization formula (1983) : (M, g^{TM}) a compact oriented Riemannian manifold. $X \in \mathcal{C}^\infty(M, TM)$ a Killing vector field and $M^X = \{x \in M : X(x) = 0\}$.
Let $\alpha \in \Omega^\bullet(M)$ such that $(d - i_X)\alpha = 0$, then

$$(1.1) \quad \int_M \alpha = \int_{M^X} \frac{\alpha}{\text{Pf}\left(\frac{-J^X + R^N}{2\pi}\right)}.$$

N be the normal bundle of M^X in M
 $J^X = \nabla^{TM} X \in \mathcal{C}^\infty(M^X, \text{End}(N))$.

Localization of Indices

- For simplicity, consider $G = S^1$.
- X : **even**-dim. compact S^1 -Spin^c manifold,
 $\text{Ind}_g(D^X \otimes E) := \text{Tr}|_{\ker(D_+^X \otimes E)}[g] - \text{Tr}|_{\ker(D_-^X \otimes E)}[g]$.
- $\{X_\alpha^{S^1}\}_\alpha$ connected comp. of $X^{S^1} = \{x \in X : gx = x, \forall g \in S^1\}$.
- N_α normal bundle of $X_\alpha^{S^1}$ in X . we fix a complex structure on N_α s. t. the weights of the S^1 -action on N_α are all positive.
 $\text{Sym}(N_\alpha^*) = 1 + \sum_{k>0} \text{Sym}^k(N_\alpha^*)$ symmetric power of N_α^* .

Theorem (Atiyah-Segal 1968, Under reform. of Liu-Ma-Zhang 2003)

$$\text{Ind}_g(D^X \otimes E) = \sum_{\alpha} \text{Ind}_g \left(D^{X_\alpha^{S^1}} \otimes \text{Sym}(N_\alpha^*) \otimes E|_{X_\alpha^{S^1}} \right)$$

as distributions on S^1 .

In particular, if $X^{S^1} = \emptyset$, $\text{Ind}_g(D^X \otimes E) = 0$.

- $\text{Ind}_g(D^X \otimes E)$ polynomial on $g, g^{-1} \in S^1$ with integral coefficients

Localization of Eta Invariants

- Y odd-dim. compact S^1 -Spin^c manifold.
- $A = \{g \in S^1 : Y^{S^1} \neq Y^g\}$ finite set.

Main result (Liu-Ma 20)

We construct explicitly $\underline{F}_{\alpha, \pm}$ as certain tensor product of the normal bundle N_α of $Y_\alpha^{S^1}$ in Y , $\chi_\alpha \in \mathbb{Z}[x]$, s.t. $\forall \underline{E}$,

$$\bar{\eta}_g(X, \underline{E}) - \left\{ \sum_{\alpha} \chi_{\alpha}(g)^{-1} \bar{\eta}_g(Y_{\alpha}^{S^1}, \underline{F}_{\alpha,+} \otimes \underline{E}|_{Y_{\alpha}^{S^1}}) - \sum_{\alpha} \chi_{\alpha}(g)^{-1} \bar{\eta}_g(Y_{\alpha}^{S^1}, \underline{F}_{\alpha,-} \otimes \underline{E}|_{Y_{\alpha}^{S^1}}) \right\}$$

as a function on $S^1 \setminus A$ is the restriction of a rational function on S^1 with **integral coefficients**, and it has no poles on $S^1 \setminus A$.

Idea of proof

- For $g \in S^1$, set $\mathbb{Q}_g := \{P(g)/P'(g) \in \mathbb{C} : P, P' \in \mathbb{Z}[x], P'(g) \neq 0\}$.
- We construct explicitly $\underline{F}_{\alpha, \pm}$ as certain tensor product of the normal bundle N_α of $Y_\alpha^{S^1}$ in Y , $\chi_\alpha \in \mathbb{Z}[x]$, s.t. $\forall \underline{E}, g \in S^1 \setminus A$,

$$\mathcal{L}(g) := \bar{\eta}_g(Y, \underline{E}) - \left\{ \sum_{\alpha} \chi_{\alpha}(g)^{-1} \bar{\eta}_g(Y_{\alpha}^{S^1}, \underline{F}_{\alpha,+} \otimes \underline{E}|_{Y_{\alpha}^{S^1}}) - \sum_{\alpha} \chi_{\alpha}(g)^{-1} \bar{\eta}_g(Y_{\alpha}^{S^1}, \underline{F}_{\alpha,-} \otimes \underline{E}|_{Y_{\alpha}^{S^1}}) \right\} \in \mathbb{Q}_g$$

One key step : inverse $\lambda_{-1}(N) = \mathbb{C} + \sum_{k>0} (-1)^k \Lambda^k(N)$.

Tools : algebraic formalism of differential K-Theory and Equivariant Bismut-Zhang embedding formula for eta invariants

- $\mathcal{L}(g) = \frac{\sum a_i g^i}{\sum b_j g^j}$. Need to show that $a_i, b_j \in \mathbb{Z}$ does not depend on g : Study carefully $\bar{\eta}_g$ as a function on S^1 .
Tools : analytic localization technique of Bismut-Lebeau

Localization of Eta Invariants

Regularity of equi. eta inv. (Liu-Ma 22)

For $g \in S^1$, $K \in \text{Lie}(S^1)$, $t \in \mathbb{R}$ and $|t|$ small, $\exists c_j(g, K) \in \mathbb{C}$, $1 \leq j \leq (\dim Y^g + 1)/2$, s. t.

$$\bar{\eta}_{ge^{tK}}(Y, \underline{E}) - \sum_{j=1}^{(\dim Y^g + 1)/2} c_j(g, K)t^{-j}$$

is an **analytic function** of t . If $g \in S^1 \setminus A$, then $c_j(g, K) = 0 \forall j$.

Goette 00 : Expansion of $\bar{\eta}_{ge^{tK}}$ as $t \rightarrow 0$ if $g = 1$, $Y^{S^1} = \emptyset$.

Theorem (Liu-Ma 20)

The function $\mathcal{L}(g)$ on $S^1 \setminus A$ is the restriction of a rational function on S^1 with integral coefficients and without poles on $S^1 \setminus A$. Thus if we know $\mathcal{L}(g)$ for a transcendental $g \in S^1 \setminus A$, we know $\mathcal{L}(g) \forall g \in S^1 \setminus A$.

Differential K-Theory

- For compact space Y , $K^0(Y)$ is the topo. K-group of Y .
- The Chern character $\text{ch} : K^0(Y) \rightarrow H^{\text{even}}(Y, \mathbb{R})$.

$$\begin{array}{ccc}
 ? & \longrightarrow & K^0(Y) \\
 \downarrow & & \downarrow \text{ch} \\
 \Omega_{\text{closed}}^{\text{even}}(Y, \mathbb{R}) & \xrightarrow{\text{de Rham}} & H^{\text{even}}(Y, \mathbb{R}).
 \end{array}$$

- $? = \widehat{K}^0(Y)$, differential K-group of Y .

History

- Differential K-theory is partly motivated by Witten 98 for R-R charge on D branes in type IIA/B superstring theory and introduced by Freed-Hopkins 00.
- It is the real analogue of the arithmetic K-theory in arithmetic algebraic geometry developed by Gillet, Soulé, etc in 1990s.

Equivalent Definitions and more :

- Hopkins-Singer 05,
- Freed-Klonoff-Lott 08, 10 ;
- Bunke-Schick 09 ;
- Simons-Sullivan 10 ;
- Tradler-Wilson-Zeinalian 13, 16 ;
- Hekmati-Murray-Schlegel-Vozzo 15 ;
- Gorokhovsky-Lott 16,

Chern-Simons Form

- $\underline{E} = (E, h^E, \nabla^E)$, curvature $R^E = (\nabla^E)^2 \in \Omega^2(Y, \text{End}(E))$,

$$\text{ch}(\underline{E}) = \text{Tr} \left[\exp \left(- \frac{R^E}{2\pi i} \right) \right] \in \Omega_{closed}^*(Y, \mathbb{R}).$$

- For $\underline{E}_1 = (E, h_1^E, \nabla_1^E)$,
 Chern-Simons form $\tilde{\text{ch}}(\underline{E}, \underline{E}_1) \in \Omega^*(Y, \mathbb{R})/\text{Im } d$ is well-defined s.t.

$$d\tilde{\text{ch}}(\underline{E}, \underline{E}_1) = \text{ch}(\underline{E}_1) - \text{ch}(\underline{E}).$$

- ∇^{π^*E} connection on π^*E (for projection $\pi : Y \times \mathbb{R} \rightarrow Y$) s.t.
 $\nabla^{\pi^*E}|_{Y \times \{0\}} = \nabla^E, \nabla^{\pi^*E}|_{Y \times \{1\}} = \nabla_1^E$, then

$$\tilde{\text{ch}}(\underline{E}, \underline{E}_1) = \int_0^1 \{\text{ch}(\pi^*E)\}^{ds} ds.$$

Definition of Differential K-Group (by Freed-Lott)

- $\underline{E} = (E, h^E, \nabla^E)$, $\phi \in \Omega^{\text{odd}}(Y, \mathbb{R})/\text{Im}d$.
- $\widehat{K}^0(Y)$, group generated by semigroup $(\underline{E}, \phi)/\sim$.
- $(\underline{E}_1, \phi_1) \sim (\underline{E}_2, \phi_2) \iff \exists \underline{E}_3$ and \exists bundle isomorphism $\Phi : E_1 \oplus E_3 \rightarrow E_2 \oplus E_3$, s.t.

$$\phi_2 - \phi_1 = \widetilde{\text{ch}}(\underline{E}_1 \oplus \underline{E}_3, \Phi^*(\underline{E}_2 \oplus \underline{E}_3)).$$

- $\widehat{K}^0(Y)$ is a ring :

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] := [\underline{E} \otimes \underline{F}, \text{ch}(\underline{E}) \wedge \psi + \text{ch}(\underline{F}) \wedge \phi - d\phi \wedge \psi].$$

- Let $\underline{\mathbb{C}}$ be the trivial complex line bundle over Y with the trivial metric and connection, $1 := [\underline{\mathbb{C}}, 0]$ is a unity for the product \cup .

Chern-Weil Theory

$$\begin{array}{ccc}
 \widehat{K}^0(Y) & \longrightarrow & K^0(Y) \\
 \downarrow \text{ch}(\underline{E}) - d\phi & & \downarrow \text{ch}([E]) \\
 \Omega_{closed}^{\text{even}}(Y, \mathbb{R}) & \xrightarrow{\text{de Rham}} & H^{\text{even}}(Y, \mathbb{R}).
 \end{array}$$

- Diagram commutes \iff Chern-Weil theory.

g -equivariant Differential K-Theory (Liu-Ma)

- Y is S^1 -compact manifold, \underline{E} is S^1 -equi. v. b. on Y .
- $g \in S^1$, $\phi \in \Omega^{\text{odd}}(Y^g, \mathbb{R})/\text{Im}d$.
- $\widehat{K}_g^0(Y)$ is the group generated by semigroup $(\underline{E}, \phi)/\sim$:
- $(\underline{E}_1, \phi_1) \sim (\underline{E}_2, \phi_2) : \exists S^1$ -equi. \underline{E}_3 and $\exists S^1$ -equi. bundle isomorphism $\Phi : E_1 \oplus E_3 \rightarrow E_2 \oplus E_3$, s.t.

$$\phi_2 - \phi_1 = \widetilde{\text{ch}}_g(\underline{E}_1 \oplus \underline{E}_3, \Phi^*(\underline{E}_2 \oplus \underline{E}_3)).$$

- $\widehat{K}_g^0(Y)$ is a ring :

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] := [\underline{E} \otimes \underline{F}, \text{ch}_g(\underline{E}) \wedge \psi + \text{ch}_g(\underline{F}) \wedge \phi - d\phi \wedge \psi].$$

- Let $\underline{\mathbb{C}}$ be the trivial complex line bundle over Y with the trivial metric and connection, $1 := [\underline{\mathbb{C}}, 0]$ is a unity for the product.

The Home of $\underline{\lambda_{-1}(N_{\alpha}^*)}^{-1}$

- $R(S^1)$ is the representation ring of S^1 . $\widehat{K}_g^0(Y)$ is $R(S^1)$ -module.
- $\widehat{K}_g^0(Y)_{I(g)}$ localization of $\widehat{K}_g^0(Y)$ at $I(g) = \{\chi \in R(S^1) : \chi(g) = 0\}$.
elements of $\widehat{K}_g^0(Y)_{I(g)} : ([\underline{E}, \phi] - [\underline{E}', \phi']) / \chi$, where $\chi(g) \neq 0$.
- $A := \{g \in S^1 : Y^g \neq Y^{S^1}\}$ finite set.

Theorem 1 (Liu-Ma 20)

For $g \in S^1 \setminus A$, $[\underline{\lambda_{-1}(N_{\alpha}^*)}, 0]$ is invertible in $\widehat{K}_g^0(Y_{\alpha}^{S^1})_{I(g)}$.

Now we sketch a proof of Theorem 1, which also gives a realization of $\underline{\lambda_{-1}(N_{\alpha}^*)}^{-1} = (\underline{F}_{\alpha,+} - \underline{F}_{\alpha,-}) / \chi$.

γ -filtration

- Pre- λ -ring Structure on $\widehat{K}^0(Y)$ s.t. $\lambda_t(\underline{E}) = \sum_{j \geq 0} \Lambda^j(\underline{E})t^j$.
- Set $\gamma_t(x) = \sum_{j \geq 0} \gamma^j(x)t^j := \lambda_{\frac{t}{1-t}}(x)$.
- For $r = \text{rk}(\underline{E})$, $\gamma^k(\underline{E} - r)$ is a finite dim. virtual vector bundle

$$\gamma^k(\underline{E} - r) = \begin{cases} \sum_{i=0}^k (-1)^{k-i} \binom{r-i}{k-i} \Lambda^i(\underline{E}) & \text{if } 0 \leq k \leq r; \\ 0 & \text{if } k > r. \end{cases}$$

$$\begin{aligned} \lambda_t(\underline{E})^{-1} &= \gamma_{\frac{t}{1+t}}(\underline{E} - r)^{-1} (1+t)^{-r} = (1+t)^{-r} \left(1 + \sum_{i=1}^r \gamma^i(\underline{E} - r) t^i (1+t)^{-i} \right)^{-1} \\ &= (1+t)^{-r} \left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k} \sum_{\substack{(n_1, \dots, n_r) \in \mathbb{N}^r, \\ \sum_{i=1}^r i \cdot n_i = k}} (-1)^{\sum_{i=1}^r n_i} \frac{(\sum_{i=1}^r n_i)!}{\prod_{i=1}^r n_i!} \prod_{i=1}^r (\gamma^i(\underline{E} - r))^{n_i} \right). \end{aligned}$$

To simplify the notations, we denote by

$$\lambda_t(\underline{E})^{-1} = (1+t)^{-r} \left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k} (P_{k,+}(\underline{E}) - P_{k,-}(\underline{E})) \right).$$

γ -filtration

$$\lambda_t(\underline{E})^{-1} = (1+t)^{-r} \left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k} (P_{k,+}(\underline{E}) - P_{k,-}(\underline{E})) \right),$$

Lemma (Liu-Ma 20)

There exists $\mathcal{N}_{r,m} > 0$, only depending on $r = \text{rk}(E)$, $m = \dim Y$, such that for any geometric triple \underline{E} on Y and $k > \mathcal{N}_{r,m}$, we have $[P_{k,+}(\underline{E}), 0] = [P_{k,-}(\underline{E}), 0] \in \widehat{K}^0(Y)$.

$$\implies \text{ch}(P_{k,+}(\underline{E})) = \text{ch}(P_{k,-}(\underline{E})) \in \Omega^{2*}(Y, \mathbb{R}).$$

Realization of $\lambda_{-1}(N_{\alpha}^*)^{-1}$

We denote by $'E$ when forgetting the group action. In the sense of $K_{S^1}^0(Y_{\alpha}^{S^1}) \simeq R(S^1) \otimes K^0(Y_{\alpha}^{S^1})$, $\lambda_{-1}(N_{\alpha}^*) \simeq \bigotimes_v \lambda_{-h-v}(\underline{'N_{\alpha,v}^*})$. Set $r_{\alpha,v} = \text{rk} N_{\alpha,v}$, $m_{\alpha} = \dim Y_{\alpha}^{S^1}$, formally,

$$\lambda_{-h-v}(\underline{'N_{\alpha,v}^*})^{-1} = \frac{h^{vr_{\alpha,v}}}{(hv-1)^{r_{\alpha,v}}} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(hv-1)^k} (P_{k,+}(\underline{'N_{\alpha,v}^*}) - P_{k,-}(\underline{'N_{\alpha,v}^*})) \right).$$

Set

$$\underline{\lambda_{-1}(N_{\alpha}^*)}_{\mathcal{N}}^{-1} = \bigotimes_v \frac{h^{vr_{\alpha,v}}}{(hv-1)^{r_{\alpha,v}}} \left(1 + \sum_{k=1}^{\mathcal{N}} \frac{(-1)^k}{(hv-1)^k} (P_{k,+}(\underline{'N_{\alpha,v}^*}) - P_{k,-}(\underline{'N_{\alpha,v}^*})) \right).$$

Theorem 2 (Liu-Ma 20)

For any $\mathcal{N} > \sup_{\alpha,v} \mathcal{N}_{r_{\alpha,v}, m_{\alpha}}$, we have

$$\left[\underline{\lambda_{-1}(N_{\alpha}^*)}, 0 \right]^{-1} = \left[\underline{\lambda_{-1}(N_{\alpha}^*)}_{\mathcal{N}}^{-1}, 0 \right] \in \widehat{K}_g^0(Y_{\alpha}^{S^1})_{I(g)}.$$

Localization of Eta Invariant ?

Question

For $\underline{\lambda}_{-1}(N_\alpha^*) := \Lambda^{\text{even}}(N_\alpha^*) - \Lambda^{\text{odd}}(N_\alpha^*)$, if $Y^g = Y^{S^1}$, $\forall \underline{E}$,

$$\bar{\eta}_g(Y, \underline{E}) \stackrel{?}{=} \sum_{\alpha} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda}_{-1}(N_\alpha^*)^{-1} \otimes \underline{E}|_{Y_\alpha^{S^1}})$$

$$\bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda}_{-1}(N_\alpha^*)_{\mathcal{N}_1}^{-1} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \stackrel{?}{=} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda}_{-1}(N_\alpha^*)_{\mathcal{N}_2}^{-1} \otimes \underline{E}|_{Y_\alpha^{S^1}}).$$

Push-forward in Differential K-Theory

Variation formula of eta invariants

If Y is odd-dim. compact S^1 -equi. Spin^c , $\exists \chi \in R(S^1)$ s. t. $\forall g \in S^1$,

$$\bar{\eta}_g(Y, \underline{E}_1) - \bar{\eta}_g(Y, \underline{E}_0) = \int_{Y_g} \text{Td}_g(TY, L) \tilde{\text{ch}}_g(\underline{E}_0, \underline{E}_1) + \chi(g).$$

Proposition (Liu-Ma 20)

Let $\mathbb{Q}_g := \{P(g)/P'(g) \in \mathbb{C} : P, P' \in \mathbb{Z}[x], P'(g) \neq 0\}$. If Y is odd-dim. S^1 -equi. Spin^c , the following push-forward map is well-defined :

$$\begin{aligned} \widehat{f}_{Y!} : \widehat{K}_g^0(Y)_{I(g)} &\rightarrow \mathbb{C}/\mathbb{Q}_g, \\ [\underline{E}, \phi]/\chi &\mapsto \chi(g)^{-1} \left(- \int_{Y_g} \text{Td}_g(TY, L) \wedge \phi + \bar{\eta}_g(Y, \underline{E}) \right) \pmod{\mathbb{Q}_g}. \\ \implies \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}_1}^{-1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) &- \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}_2}^{-1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \in \mathbb{Q}_g. \end{aligned}$$

Localization of Eta Invariant ?

Question

If $Y^g = Y^{S^1}$, i.e., $g \in S^1 \setminus A$, for \mathcal{N} large, $\forall \underline{E}$,

$$\bar{\eta}_g(Y, \underline{E}) - \sum_{\alpha} \bar{\eta}_g(Y_{\alpha}^{S^1}, \underline{\lambda}_{-1}(N_{\alpha}^*)_{\mathcal{N}}^{-1} \otimes \underline{E}|_{Y_{\alpha}^{S^1}}) \in \mathbb{Q}_g?$$

Differential Atiyah-Hirzebruch Direct Image

Equivariant Bismut-Zhang embedding formula (B-Z 93, Liu 18)

Consider the embedding $\iota : Y^{S^1} \rightarrow Y$. For S^1 -equi. $\underline{\mu}$ over Y^{S^1} , we can constr. the geometric version of equi. A-H direct image $\underline{\xi}_{\pm} = \hat{\iota}_!(\underline{\mu})$. Moreover, there exists $\chi' \in R(S^1)$, s.t. for any $g \in S^1 \setminus A$,

$$\sum_{\alpha} \bar{\eta}_g(Y_{\alpha}^{S^1}, \underline{\mu}) = \bar{\eta}_g(Y, \underline{\xi}_+) - \bar{\eta}_g(Y, \underline{\xi}_-) + \chi'(g).$$

Theorem 3 (Liu-Ma 20)

$\iota : Y^{S^1} \rightarrow Y$ induces a pullback map $\hat{\iota}^* : \widehat{K}_g^0(Y)_{I(g)} \rightarrow \widehat{K}_g^0(Y^{S^1})_{I(g)}$, which is an isomorphism. Moreover, the following diagram commutes :

$$\begin{array}{ccc} \widehat{K}_g^0(Y^{S^1})_{I(g)} & \xleftarrow{[\lambda_{-1}(N^*), 0]^{-1} \cup \hat{\iota}^*} & \widehat{K}_g^0(Y)_{I(g)} \\ & \searrow \widehat{f}_{Y^{S^1}!} & \swarrow \widehat{f}_Y! \\ & \mathbb{C}/\mathbb{Q}_g & \end{array}$$

Localization of Eta Invariants

- Recall that $\widehat{f}_{Y!}([E, \phi]) = -\int_{Yg} \text{Td}_g(TY, L) \wedge \phi + \bar{\eta}_g(Y, \underline{E})$.

$$\begin{array}{ccc}
 \widehat{K}_g^0(Y^{S^1})_{I(g)} & \xleftarrow{[\lambda_{-1}(N^*), 0]^{-1} \cdot i^*} & \widehat{K}_g^0(Y)_{I(g)} \\
 \searrow \widehat{f}_{Y^{S^1}!} & & \swarrow \widehat{f}_{Y!} \\
 & \mathbb{C}/\mathbb{Q}_g &
 \end{array}$$





$$[\lambda_{-1}(N_\alpha^*), 0]^{-1} = [\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}, 0] = [0].$$

Theorem (Liu-Ma 20)

If Y is odd-dim. S^1 -equi. Spin^c , for $g \in S^1 \setminus A$, for \mathcal{N} large, $\forall \underline{E}$,

$$\mathcal{L}(g) := \bar{\eta}_g(Y, \underline{E}) - \sum_{\alpha} \bar{\eta}_g(Y_\alpha^{S^1}, \lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \in \mathbb{Q}_g.$$

Reference

-  B. Liu, *Real embedding and equivariant eta forms*, Math. Z. **292** (2019), 849–878.
-  B. Liu and X. Ma, *Differential K-theory, η -invariants and localization*, C. R. Math. Acad. Sci. Paris, **357** (2019), no. 10, 803–813.
-  B. Liu and X. Ma, *Differential K-theory and localization formula for η -invariants*, Invent. Math. **222** (2020), 545–613.
-  B. Liu and X. Ma, *Comparison of two equivariant eta forms*, Adv. Math., **404** (2022), 108163, 76pp.

*Thank you.
Happy birthday, Paolo!*