Higher Orbital Integral and APS Index Theorem

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Manifolds with boundary

- Y = compact Riemannian manifold with product metric near its boundary.
- $\widetilde{Y}$ = the associated b-manfiold
  = the associated manifold with cylindrical ends.
- D_Y= twisted Spin^c-Dirac operator on Y.
- $D_{\partial Y}$ is L^2-invertible $\Rightarrow D_{\widetilde{Y}}$ is Fredholm.
- The APS index

$$\mathrm{Ind}_{APS}(D_Y) = \mathrm{Ind}(D_{\widetilde{Y}}) \in \mathbb{Z}.$$

The APS index theorem

Theorem (Atiyah-Patodi-Singer)

$$\mathrm{Ind}_{APS}(D_Y) = \int_Y AS(D_Y) - \frac{1}{2}\eta(D_{\partial Y}),$$

Here

$$\eta(D_{\partial Y}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}\left(D_{\partial Y} \exp\left(-tD^2_{\partial Y}\right)\right) \frac{dt}{\sqrt{t}}$$

is the eta invariant measuring the spectral asymmetry of $D_{\partial Y}$.

Geometric assumptions

- G = connected, linear reductive Lie group
  = connected subgroup of $GL(n, \mathbb{R})$, stable under transpose.
- K = maximal compact subgroup of G.
- Y = smooth manifold with proper, cocompact action of G.
- Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
- Abel's slice theorem: $Y = G \times_K Z$. Here Z is a compact K-manifold with boundary.
- $TY \cong G \times_K (\mathfrak{p} \oplus TZ)$.
- We assume the G-invariant metric on Y is induced from a K-invariant metric on Z and a K-invariant metric on $\mathfrak{p}$.
- In this case, the Dirac operator decomposes

$$D_Y = D_{G/K} \otimes 1 + 1 \otimes D_Z.$$

Higher index

Theorem (Piazza-Posthuma-S-Tang)

Suppose that $D_{\partial Y}$ is L^2-invertible. Then

- *The smoothing kernel κ_t of $e^{-tD_Y^2}$ lies in $\mathcal{A}_G^\infty(Y)$.*
- *There is a smooth index class*

$$\mathrm{Ind}_\infty(D_Y) \in K\left(\mathcal{A}_G^\infty(Y)\right) \cong K(C_r^*G) \cong K\left(\mathcal{C}(G)\right).$$

The algebra

$$\mathcal{A}_G^\infty(Y) \cong \left(\mathcal{C}(G) \otimes \Psi^{-\infty}(Z)\right)^{K\times K},$$

where

- $\mathcal{C}(G)$ = Harish-Chandra Schwartz algebra of G.
- $\Psi^{-\infty}(Z)$ = some algebra of smoothing operators on the compact manifold Z defined using Melrose's b-calculus.

Why higher index?

If $G = \mathbb{R}^2$ acts on $Y = \mathbb{R}^2$ by translation, then $\ker(D)$ and $\ker(D^*)$ are all trivial. The L^2-index is trivial but $\mathrm{Ind}_\infty(D)$ is non-trivial in $K(\mathcal{C}(G))$.

A question by Connes-Moscovici: how can we exact a non-zero number from $\mathrm{Ind}_\infty(D)$?

By Fourier transform, $(\mathcal{C}(\mathbb{R}^2), *) \cong (\mathcal{S}\left(\widehat{\mathbb{R}^2}\right), \cdot)$. The homology

$$H_\bullet\left(\widehat{\mathbb{R}^2}\right) = \begin{cases} \mathbb{R}, & \bullet = 2 \\ 0, & \text{otherwise} \end{cases}$$

and is generated by a degree 2 differential current

$$\Psi(\widehat{f_0}, \widehat{f_1}, \widehat{f_2}) = \int_{\mathbb{R}^2} \widehat{f_0}\, d\widehat{f_1} d\widehat{f_2}.$$

Cyclic cocycles on $\mathbb{R}^2$

Define a function $c\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$c(x, y) = x_1 y_2 - x_2 y_1$$

Define a cocycle on $C(\mathbb{R}^2)$ by

$$\Phi(f_0, f_1, f_2) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} c(x, y) \cdot f_0(-x - y) f_1(x) f_2(y) \, dx dy.$$

One can check

$$\Phi(f_0 * f_1, f_2, f_3) - \Phi(f_0, f_1 * f_2, f_3) + \Phi(f_0, f_1, f_2 * f_3) - \Phi(f_3 * f_0, f_1, f_2) = 0$$

and

$$\Phi(f_2, f_0, f_1) = -\Phi(f_0, f_1, f_2).$$

Thus Φ is a degree 2 cyclic cocycle on $C(\mathbb{R}^2)$ and the higher index pairing

$$\langle \Phi, \mathrm{Ind}_\infty(D) \rangle = 1.$$

General cases

For general reductive Lie group,

$$C(G) \sim \mathcal{S}\left(\widehat{G}\right) \sim \mathcal{S}\left(\sqcup\left(\mathbb{R}^n / W\right)\right),$$

where W is some finite group (I am lying).

Let $P = MAN$ be a parabolic subgroup of G. The choice of P determines some choices of the connected components of $\widehat{G}$.

For each P, we can associate it with some differential currents, so that we can define higher index pairing. This generalizes the L^2-trace on $C(G)$.

Theorem (S-Tang)

We can define higher degree cyclic cocycles for P on $C(G)$.

Higher orbital integral (delocalized)

- g = semi-simple element in G, and Z_g is its centralizer.
- The orbital integral is define by

$$\tau_g(f) = \int_{G/Z_g} f(hgh^{-1})d(hZ_g), \quad f \in \mathcal{C}(G).$$

- The orbital integral induces a continuous trace

$$\tau_g^Y : K(\mathcal{C}(G)) \to \mathbb{C}.$$

- If $g = e$, then τ_e^Y is the L^2-trace.
- If P is a parabolic subgroup of G, the higher orbital integral defines a higher pairing

$$\tau_{P,g}^Y : K(\mathcal{C}(G)) \to \mathbb{C}.$$

0-degree delocalized APS index theorem

Let Y be a cocompact proper G-manifold with boundary and g be a semi-simple element in G.

Theorem (Piazza-Posthuma-S-Tang)

If $D_{\partial Y}$ is L^2-invertible, then

$$\left\langle \tau_g^Y, \mathrm{Ind}_\infty(D_Y) \right\rangle = \int_{Y^g} AS(D_Y) - \frac{1}{2}\eta_g(D_{\partial Y}),$$

where the delocalized eta invariant

$$\eta_g(D_{\partial Y}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^{\partial Y}\left(D_{\partial Y} \exp\left(-tD_{\partial Y}^2\right)\right) \frac{dt}{\sqrt{t}}.$$

Similar results were also obtained by Hochs-Wang-Wang under the assumption that G/Z_g is compact.

Higher degree delocalized APS index theorem

- $P = MAN$ is a cuspidal parabolic subgroup.
- The group AN acts freely on Y.
- g = a semi-simple element in M.

Theorem (Piazza-Posthuma-S-Tang)

If $D_{\partial Y}$ is L^2-invertible, then

$$\left\langle \tau^{Y}_{P,g}, \mathrm{Ind}_\infty(D_Y) \right\rangle = \int_{(Y/AN)^g} AS(D_{Y/AN}) - \frac{1}{2}\eta_g(D_{\partial Y/AN}),$$

where the delocalized eta invariant

$$\eta_g(D_{\partial Y/AN}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^{\partial Y/AN}\left(D_{\partial Y/AN} \exp\left(-tD^2_{\partial Y/AN}\right)\right) \frac{dt}{\sqrt{t}}.$$

For the index pairing with higher cyclic cocycles from group cohomology, the corresponding APS theorem was proved by Piazza-Posthuma.

Eta invariant

- X = cocompact G-proper manifold without boundary.
- g = semi-simple element of G.

Theorem (Piazza-Posthuma-S-Tang)

The delocalized eta invariant

$$\eta_g(D_X) = \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^X \left(D_X \exp\left(-tD_X^2\right)\right) \frac{dt}{\sqrt{t}}$$

is convergent.

Notice that we are not assuming L^2-invertibility nor a spectral gap at 0 for the operator D_X. For $X = G/K$, this result was proved by Moscovici-Stanton and Bismut.

When G is a discrete group, the well-definedness of eta invariant has been studied by many people.

Spectrum gap: global perturbation

- Assume that there exists $\delta > 0$ such that
$$\mathrm{Spec}(D_{\partial Y}) \cap (-\delta, \delta) = \{0\}.$$
- Take $\theta \in (0, \delta)$ and consider perturbed operator
$$D_{\partial Y}(\theta) = D_{\partial Y} + \theta.$$
- $D_{\partial Y}(\theta)$ is L^2-invertible.
- Spectrum gap can only occur when
  - G is of equal rank.
  - $\ker(D_{\partial Y})$ contains only discrete series representation of G. Equivalently, this requires that $\ker(D_{\partial Z})$ contains only "regular" K-representations.

APS index theorem for global perturbation

We apply the APS index theorem to $D_{\partial Y}(\theta)$ and take $\theta \to 0$.

Theorem (Atiyah-Patodi-Singer)

If Y is compact,

$$\mathrm{Ind}_{APS}(D_Y) = \int_Y AS(D_Y) - \frac{1}{2}\left(\eta(D_{\partial Y}) + \dim(\ker(D_{\partial Y})\right).$$

Theorem (Piazza-Posthuma-S-Tang)

$$\left\langle \tau_g^Y, \mathrm{Ind}_\infty(D_Y)\right\rangle = \int_{Y^g} AS(D_Y) - \frac{1}{2}\left(\eta_g(D_{\partial Y}) + \tau_g^{\partial Y}(\Pi_{\ker D_{\partial Y}})\right).$$

Perturbation on the slice

- The Dirac operator decomposes
$$D_{\partial Y} = D_{G,K} \otimes 1 + \epsilon \otimes D_{\partial Z}$$
on
$$L^2(\partial Y, E|_{\partial Y}) \cong \left[L^2(G) \otimes S_{\mathfrak{p}} \otimes L^2(\partial Z, E|_{\partial Z})\right]^K,$$
and ϵ is the $\mathbb{Z}_2$-grading on $S_{\mathfrak{p}}^{\pm}$.
- Because ∂Z is a compact manifold,
$$\mathrm{Spec}(D_{\partial Z}) \cap (-\delta, \delta) = \{0\}.$$
- For $\theta \in (0, \delta)$, we consider
$$D_{\partial Y}(\theta) = D_{G,K} \otimes 1 + \epsilon \otimes (D_{\partial Z} + \theta) \neq D_{\partial Y} + \theta$$
- $D_{\partial Y}(\theta)$ is L^2-invertible.

APS index theorem for slice perturbation

Let $P = MAN$ be a cuspidal parabolic subgroup and $g \in M$ be a semi-simple element.

Theorem (Piazza-Posthuma-S-Tang)

$$\left\langle \tau^{Y}_{P,g}, \mathrm{Ind}_{\infty}(D_Y) \right\rangle = \int_{(Y/AN)^g} AS(D_Y) - \frac{1}{2}\eta_g(D_{\partial Y/AN})$$
$$-\frac{1}{2}\langle D\text{-}Ind^{M}_{M\cap K}(\ker D_{\partial Z}), \tau^{M}_{g}\rangle.$$

Here the map

$$\text{D-Ind}^{M}_{M\cap K} : \text{Repn of } K \cap M \to K(C^{*}_{r}M)$$

is the Dirac induction used in the Connes-Kasparov theorem.